Homework due Monday, November 20.

1. a) Show that if $f(x) = \dfrac{e^{-|x|}}{\sqrt{x}}$, then $f(x) \in L^1(\mathbb{R})$ and $f(x) \notin L^2(\mathbb{R})$.

   b) Show that if $g(x) = \dfrac{1}{\sqrt{1+x^2}}$, then $g(x) \notin L^1(\mathbb{R})$ and $g(x) \in L^2(\mathbb{R})$.

2. Let $T: \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation defined by $T[\vec{v}] = A\vec{v}$, where A is an $n \times n$ matrix with complex entries. Also let $\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \overline{\vec{w}} = \sum_{i=1}^{n} v_i \overline{w_i}$ be the standard inner product on $\mathbb{C}^n$.

   a) Define $T^*: \mathbb{C}^n \to \mathbb{C}^n$ by $\langle T^*[\vec{v}], \vec{w} \rangle = \langle \vec{v}, T[\vec{w}] \rangle \ \forall\, v, w \in \mathbb{C}^n$.
   Prove that $T^*[\vec{v}] = \overline{A^T}\vec{v}$

   b) Suppose $T: \mathbb{C}^n \to \mathbb{C}^n$ is such that $\langle T[\vec{v}], T[\vec{w}] \rangle = \langle \vec{v}, \vec{w} \rangle \ \forall\, \vec{v}, \vec{w} \in \mathbb{C}^n$.
   Prove that $A \cdot \overline{A^T} = I$, where I is the identity matrix.