SOCR Math Problems

(1) Here is a problem: Solve the equation $x^2 - x = 12$. [Do not assume knowledge of the quadratic formula in this problem.]
   (a) Just solve the problem.
   (b) Write the problem and its solution as a rigorous "If ... , then ..." statement.
   (c) Prove your statement.

(2) Consider the equation $ax^2 + bx + c = 0$, with a, b, c real numbers.
   (a) Make a rigorous statement that tells you how many real number solutions x there are to the equation.
   (b) Prove your statement.

(3) Prove or disprove each of the following, if a is assumed to be a complex number.
   (a) If $\sqrt{a-4} = a - 6$, then $a = 5$ or $a = 8$.
   (b) If $a = 5$ or $a = 8$, then $\sqrt{a-4} = a - 6$.
   (c) The solution set to the equation $\sqrt{a-4} = a - 6$ is $\{8\}$.

(4) Prove, using the properties alone, that cancellation works when using complex numbers. Specifically, suppose $a, b, c \in \mathbb{C}$ and $a \neq 0$.
   (a) Prove that if $ab = ac$ then $b = c$.
   (b) Prove or disprove that the hypothesis $a \neq 0$ is necessary.

(5) When we write $a + b + c$, do we mean $(a + b) + c$ or $a + (b + c)$? Why?

(6) Prove, using the properties alone, that if $a, b, c, d \in \mathbb{C}$, then $a\,(b + c + d) = ab+ac+ad$.

(7) Prove, using the properties alone, that if $x \in \mathbb{C}$ and $a_j \in \mathbb{C}$ for $j = 1$ to n, then

$$x \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} x a_j.$$

(8) Prove the following using the properties alone.
   (a) If $a, b, c, d \in \mathbb{C}$, then $(a + b)(c + d) = ac + ad + bc + bd$.
   (b) If $a_j \in \mathbb{C}$ for $j = 1$ to n and $b_k \in \mathbb{C}$ for $k = 1$ to m, then

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{m} b_k\right) = \sum_{j=1}^{n}\left(\sum_{k=1}^{m} a_j b_k\right).$$

(9) Prove the following properties of complex numbers. Let x, y be any real numbers. The complex conjugate $\overline{x + yi}$ is defined by the equation $\overline{x + yi} = x - yi$. The absolute value $|x + yi|$ is defined as $|x + yi| = \sqrt{x^2 + y^2}$. Also, $\mathrm{Re}\,(x + iy) = x$, and $\mathrm{Im}\,(x + iy) = y$.
   (a) For all $z, w \in \mathbb{C}$, $\overline{z + w} = \overline{z} + \overline{w}$.
   (b) For all $z, w \in \mathbb{C}$, $\overline{zw} = \overline{z}\;\overline{w}$.
   (c) For all $z, w \in \mathbb{C}$, $\overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}}$.
   (d) For all $z \in \mathbb{C}$, $|\overline{z}| = |z|$.
   (e) For all $z \in \mathbb{C}$, $|\mathrm{Im}\,(z)| \leq |z|$; equality occurs only when ________. [Fill in the blank, and prove.]
   (f) For all $z, w \in \mathbb{C}$, $|zw| = |z|\,|w|$.
   (g) For all $z, w \in \mathbb{C}$, $|z + w|^2 = |z|^2 + |w|^2 + 2\mathrm{Re}\,(z\overline{w})$.
   (h) For all $z, w \in \mathbb{C}$, $|z + w| \leq |z| + |w|$, and equality occurs only when z and w lie on the same ray through the origin in $\mathbb{C}$.

(i) For all $z, w \in \mathbb{C}$, $|z+w| \geq |z| - |w|$, and equality occurs only when _________. [Fill in the blank, and prove.]

(j) For all $z, w \in \mathbb{C}$, $|zw| \leq \frac{1}{2}\left(|z|^2 + |w|^2\right)$.

(k) For all $z, w \in \mathbb{C}$ and c a positive real number, $|zw| \leq \frac{1}{2c}|z|^2 + c|w|^2$.

(10) Describe the graph of the equation $x^2 - 2x + y^2 = 0$ in terms of the complex variable $z = x + iy$. Graph it.

(11) Suppose that A, B, C are real constants, and let $\ell = \{(x, y) : Ax + By = C\}$. Give an equivalent description of ℓ in terms of the complex variable $z = x + iy$.

(12) Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial with real number coefficients, with $a_n \neq 0$. Prove that if z_0 is a root of $p(x)$, then $\overline{z_0}$ is also a root of $p(x)$.

(13) Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial with real number coefficients, with $a_n \neq 0$. Prove that $p(x)$ can be factored as a product of polynomials with real number coefficients such that the degree of each factor is at most 2. [You may assume the fundamental theorem of algebra for this problem, which states that every polynomial $q(z) = \sum_{s=0}^{m} c_s z^s$ with coefficients in $\mathbb{C}$ and $c_m \neq 0$ can be factored as $q(z) = c_m \prod_{r=1}^{m}(z - z_r)$, where $z_1, ..., z_m$ are the (complex) roots of $q(z)$, counted with multiplicities.]

(14) If $z = re^{i\theta}$ and $w = se^{i\varphi}$ (with $r, s, \theta, \varphi \in \mathbb{R}$, $r > 0, s > 0$), prove that $s - r \leq |z - w| \leq r + s$.

(15) Using the definition of e^z with $z \in \mathbb{C}$, prove that $e^{z+w} = e^z e^w$. [You may use facts about real exponentials in this problem.]

(16) Solve the following equations for $z \in \mathbb{C}$.

(a) $z^2 + 64 = 0$

(b) $z^3 + 64 = 0$

(c) $z^4 + 64 = 0$

(d) $z^6 + 64 = 0$

(e) $z^6 - 64 = 0$

(f) $z^2 + z + 1 = 0$

(g) $z^3 + z^2 + z + 1 = 0$

(h) $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$

(i) $2^z = 4$

(j) $2^z = -4$

(k) $2^z = 0$

(17) By substituting in the Taylor series, find a simple formulas for each of the following, where $x, y \in \mathbb{R}$:

(a) $\cos(x + iy)$

(b) $\sin(x + iy)$

(c) $\cosh(x + iy)$ [Note that $\cosh(x) = \frac{e^x + e^{-x}}{2}$.]

(d) $\sinh(x + iy)$ [Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$.]

(e) 3^{x+iy}

(18) Prove that the sum of two rational numbers is always a rational number.

(19) Prove that the quotient of two nonzero rational numbers is always a nonzero rational number.

(20) Prove that $\sqrt{45}$ is irrational.

(21) Prove that if n is an integer, then $\sqrt{n}$ is a rational number if and only if n is a perfect square.

(22) Consider the equation $ax^2 + bx + c = 0$, where x is assumed to be a complex number and a, b, c are fixed integers.
   (a) Prove or disprove that x is always a rational number.
   (b) Prove or disprove that if one solution to $ax^2 + bx + c = 0$ is an integer, then all solutions of the equation are integers.
   (c) Prove or disprove that if one solution to $ax^2 + bx + c = 0$ is a rational number, then all solutions of the equation are rational numbers.

(23) Prove that $e^x e^y = e^{x+y}$ using the Taylor series definition only. For this problem, you may assume that the convergence issues do not pose a problem, so that you can multiply the series and group terms as desired.

(24) Prove or disprove the following statements:
   (a) If S and T are finite sets, and if there exists a 1-1 function $f : S \to T$ that is not onto, then there does not exist a 1-1 function $g : T \to S$.
   (b) If S and T are any sets, and if there exists a 1-1 function $f : S \to T$ that is not onto, then there does not exist a 1-1 function $g : T \to S$.

(25) Prove that if $F : U \to V$ is an injective function, then there exists a subset P of V such that $|U| = |P|$.

(26) Prove that the union of two countable sets is countable.

(27) Prove that the union of a countable set and an uncountable set is uncountable.

(28) Prove that the union of two uncountable sets is uncountable.

(29) Prove that if the set S is uncountable, there exists a set $U \subseteq S$ such that U is countable.

(30) Recall that the set difference $A \setminus B$ is defined to be

$$A \setminus B = \{x \in A : x \notin B\}.$$

   (a) If $A = \{1, 2, 5\}$, $B = \{4, 5\}$, compute $A \setminus B$ and $B \setminus A$.
   (b) Prove that if V is a countable set and if $U \setminus V$ is a countable set, then U is a countable set.
   (c) Prove that if U is an uncountable set and V is a countable set, then $U \setminus V$ is uncountable.

(31) Show that there exists a bijection from the set of irrational numbers to $\mathbb{R}$.

(32) Prove or disprove:
   (a) If A and B are two uncountable sets, then $A \cap B$ is uncountable.
   (b) If A and B are two uncountable sets, then $A \cap B$ is not uncountable.
   (c) If A and B are two countable sets, then $A \cap B$ is countable or finite.

(33) Find an explicit bijection from $\mathbb{R}$ to $\mathbb{R} \setminus \{0\}$.

(34) Let $(0, 1)$ denote the open interval $\{x \in \mathbb{R} : 0 < x < 1\}$. Find a bijection from $\mathbb{R}$ to $(0, 1)$.

(35) Let $[0, 1]$ denote the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Find a bijection from $\mathbb{R}$ to $[0, 1]$.

(36) Prove that there exists a sequence of open intervals $(I_1, I_2, ...)$ in $\mathbb{R}$ such that for all $j \in \mathbb{N}$, $I_j = (a_j, b_j)$ with $a_j < b_j$ such that the total length of the intervals is less than 0.001 and such that $\mathbb{Q} \subseteq \bigcup_{j=1}^{\infty} I_j$. That is, the length of $I_j = (a_j, b_j)$ is $\ell(I_j) = b_j - a_j$,

and the total length is defined to be

$$L = \sum_{j=1}^{\infty} (b_j - a_j),$$

even though some of the intervals might overlap.

(37) Let $h : \mathbb{Z} \to \mathbb{Z}$ be defined by $h(x) = x^3$.
  (a) Prove or disprove that h is injective.
  (b) Prove or disprove that h is surjective.

(38) Let $\phi : \mathbb{N} \to \mathbb{N}$ be defined by $\phi(n) =$ the number of digits in the binary expansion of n.
  (a) Prove or disprove that ϕ is injective.
  (b) Prove or disprove that ϕ is surjective.

(39) Let $\mathbb{Z} \times \mathbb{Z} = \{(x, y) : x, y \in \mathbb{Z}\}$. Prove that $\mathbb{Z} \times \mathbb{Z}$ is countable.

(40) Do the following computations.
  (a) Express 784 as a base-2 number.
  (b) Express 784 as a base-5 number.
  (c) Express 23.4 as a base-2 number.
  (d) Express 23.4 as a base-3 number.
  (e) Find (in base 5) $1234_5 + 403_5$.
  (f) Write $\frac{2}{3}$ as a base 3 number.
  (g) Write $\frac{2}{3}$ as a base 4 number.
  (h) Find (in base 2) $(1011_2)(1010010_2)$
  (i) Find $\frac{10011_2}{1111_2}$ in base 2.
  (j) Find $\frac{10011_2}{1111_2}$ in base 10.
  (k) Find $\frac{20102_3}{2201_3}$ in base 3.
  (l) Write $0.2022\overline{022}_3$ as a base 3 fraction.

(41) Let $C_0 = [0, 1]$ and for $n \geq 1$, $C_n = C_{n-1}$ with the middle thirds of the subintervals removed, as in the construction of the Cantor set $C = \bigcap_{n=1}^{\infty} C_n$. Prove that if $y \in [0, 1] \setminus C$, then there exists a positive integer k such that $y \in C_{k-1}$ and $y \notin C_k$.

(42) Let $M = \{x \in [0, 1] : x$ can be represented as a decimal number with no 4's in its decimal expansion $\}$.
  (a) Prove that M has measure zero. That is, for every (small) $\varepsilon > 0$, there is a collection of intervals $\{I_j\}$ such that $\sum_j \text{length}(I_j) < \varepsilon$ and $M \subseteq \bigcup_j I_j$. [Note: it is okay if you use closed or open intervals.]
  (b) Prove that M is uncountable.

(43) Prove that the union of a finite number of countable sets is countable. [Hint: induction.]

(44) Let S_n be the sum of the first n multiples of 8. Find a closed formula for S_n, and prove it using induction.

(45) Prove that for every integer $n \geq 10$, $2^n > n(n+1)(n-1)$.

(46) Prove using induction that for every positive integer n and every real number x, $1 - x^n = (1 - x) \sum_{m=0}^{n-1} x^m$.

(47) Prove that for every even integer n, $7^n - 1$ is divisible by 8.
  (a) using induction
  (b) using modular arithmetic

(48) The populations of various fictitious states are listed below, and delegates from the states are to be apportioned for a legislature of size 200.

state	Aardvark	Bubba	Cookie	Doggie	Elephant	Fun-ction	Googoo
population	241,111	32,000	700,000	104,444	212,222	110,000	111,111

  (a) Compute the delegate quota for each state. What divisor is used?
  (b) Using a spreadsheet, compute the delegate apportionments for each state, using Hamilton's method, Jefferson's method, Webster's method, Adam's method, and Hill's method. Indicate which divisors were used in your calculation. Send me or print out a copy of the spreadsheet.
  (c) Are there any violations of quota in any of the methods?

(49) Let $S = \{a, b, c, d, e\}$ be a set with operation $*$ that satisfies the following multiplication table.

For example, to look up $a * e$, you find the a row and the e column, and you get $a * e = d$.

$*$	a	b	c	d	e
a	b	c	a	e	d
b	d	e	b	c	a
c	a	b	c	d	e
d	e	a	d	b	c
e	c	d	e	a	b

  (a) $(S, *)$ satisfies the identity property. Find the identity for this set and operation.
  (b) Prove or disprove that $(S, *)$ is a group.
  (c) Prove or disprove that $(S, *)$ satisfies the commutative property.

(50) Consider each of the following subsets of $\mathbb{Z}_{12}$. Which are subgroups? Justify your answers.
  (a) $\{0\}$
  (b) $\{4, 8\}$
  (c) $\{0, 4, 8\}$
  (d) $\{0, 5, 7\}$
  (e) $\{0, 3, 6, 9\}$
  (f) $\{0, 1, 2, 4, 5, 7, 8, 10, 11\}$

(51) Consider the set $H = (0, \infty)$ with multiplication as the operation.
  (a) Show that H is a group.
  (b) Prove or disprove that $\mathbb{Q} \cap H$ is a subgroup of H.
  (c) Prove or disprove that $(0, 10)$ is a subgroup of H.
  (d) Prove or disprove that $\mathbb{Z} \cap H$ is a subgroup of H.
  (e) Find all possible finite subgroups of H.

(52) Find all the subgroups of each of the following groups.
  (a) $\mathbb{Z}_3$
  (b) $\mathbb{Z}_4$
  (c) $\mathbb{Z}_6$
  (d) $U(\mathbb{Z}_8) = \{x \in \mathbb{Z}_8 : \gcd(8, x) = 1\}$ with multiplication mod8 as the operation.
  (e) $U(\mathbb{Z}_6)$
  (f) $U(\mathbb{Z}_7)$

(53) Find all the right cosets of the subgroup H in the group G.
  (a) $G = \mathbb{Z}_{30},\ H = \{0, 5, 10, 15, 20, 25\}$
  (b) $G = U(\mathbb{Z}_{12}),\ H = \{1, 5\}$
  (c) $G = U(\mathbb{Z}_{10}),\ H = \langle 7 \rangle =$ subgroup generated by 7.

(54) Let H be a subgroup of a group G with multiplication as the group operation. Prove that
  (a) If $a \in H$, then $aH = H$.
  (b) If $a \in G$ and $aH = H$, then $a \in H$.
  (c) If $a, b \in G$ and $aH = bH$, then there exists $k \in H$ such that $ak = b$.
  (d) If $a, b \in G$ and $aH \cap bH \neq \varnothing$, then $aH = bH$.

(55) Consider

$$\begin{aligned} D_4 &= \text{the group of symmetries of the square } [-1,1] \times [-1 \times 1] \\ &= \{e, R_{90}, R_{180}, R_{270}, F_x, F_y, F_{y=x}, F_{y=-x}\}, \end{aligned}$$

where e is the identity, $R_n =$ rotation by $n°$counterclockwise, and $F_x =$ reflection across the x-axis, $F_y =$ reflection across the y-axis, $F_{y=x}$ is reflection across the line $y = x$, and $F_{y=-x}$ is reflection across the line $y = -x$.
  (a) Calculate each of the following:
    (i) $R_{90} \circ F_y$
    (ii) $F_{y=x} \circ R_{270} \circ F_x$
  (b) Prove that $\{e, F_{y=x}\}$ is a subgroup of D_4.
  (c) Show that $\{e, F_y, F_{y=x}\}$ is not a subgroup of D_4.
  (d) Find all the left cosets of $\{e, F_{y=x}\}$ in D_4.
  (e) Find all the right cosets of $\{e, F_{y=x}\}$ in D_4.

(56) Let $Q = \{1, -1, i, -i, j, -j, k, -k\}$ be the quaternion group with relations $i^2 = -1 = j^2 = k^2$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.
  (a) Prove or disprove that $\{1, i\}$ is a subgroup of Q.
  (b) Prove or disprove that $\{1, i, -1, -i\}$ is a subgroup of Q.
  (c) Find all the left cosets of $\{1, -1\}$ in Q.

(57) Below are several examples of permutations. For each each example, express it as a product of disjoint cycles, and then compute the order of the permutation.
  (a) $\alpha = (23)(13)(12)$.
  (b) $\beta(1) = 4,\ \beta(2) = 2,\ \beta(3) = 1,\ \beta(4) = 3$.
  (c) $\gamma \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \\ 3 \end{pmatrix}$.
  (d) $\delta = (134)(23567)(12)$.
  (e) $\varepsilon = (12)(234175)(12)$.

(58) Let H and K be two subgroups of the group $(G,*)$. Prove that $H\cap K$ is a subgroup of G.

(59) Let $H=\{e,(123),(132)\}$, $K=\{e,(12)\}$.
  (a) Prove that H is a subgroup of S_3.
  (b) Prove that K is a subgroup of S_3.
  (c) Find all the left cosets of H in S_3.
  (d) Find all the left and right cosets of K in S_3.

(60) Prove that there is no element of order 6 in S_4.

(61) Simplify the following, using modular arithmetic facts.
  (a) $47^{5678342} \bmod 3$
  (b) $47^{5678342} \bmod 5$
  (c) $7897943^{795494548} \bmod 10$
  (d) $7897943^{795494549} \bmod 10$
  (e) $82^{678932} \bmod 11$
  (f) $(11^{100}+12^{100}+13^{100}) \bmod 8$

(62) Find a closed form formula for each sequence below.
  (a) $x_n=2+x_{n-1};\ x_1=5$
  (b) $x_1=2;\ x_p=2px_{p-1}$
  (c) $a_0=3,\ a_{k+1}=a_k+\frac{1}{2^k}$
  (d) $b_n=b_{n-1}+n;\ b_1=6$

(63) Solve the following inequalities. Graph your solution on the number line.
  (a) $|t+2|<3$
  (b) $|\sqrt{y}-2|<1$
  (c) $\left|\dfrac{x+2}{x}-3\right|<\dfrac{1}{2}$
  (d) $\left|h^3+8\right|<1$

(64) Find the limit of each sequence, and prove the result is correct.
  (a) $\lim\limits_{k\to\infty}\dfrac{k^2+1}{k^2}$
  (b) $\left(\dfrac{2}{3-j}\right)_{j\geq 4}$
  (c) $\lim\limits_{n\to\infty}\dfrac{5n+5n^2}{(n+1)(n+2)}$
  (d) $\lim\limits_{m\to\infty}\left(\left(1+\dfrac{1}{m}\right)^2+2\right)$
  (e) $\lim\limits_{s\to\infty}\sqrt{4-\dfrac{3}{s}}$
  (f) $\left(\dfrac{2(-1)^p}{4+2p}\right)_{p\geq 1}$

(65) Prove that the limit of each sequence below does not exist.

(a) $\lim_{k\to\infty} \dfrac{(-1)^k}{10}$

(b) $\lim_{m\to\infty} \dfrac{m^2+1}{m}$

(c) $\lim_{p\to\infty} \sqrt{p+1}$