# Graduate student problem.

#4.3.11 a) Given $y \in \mathbb{R}$ and given $\varepsilon > 0$ choose $\delta = \varepsilon$, then $\forall x \in \mathbb{R}$, $|x-y| < \delta \Rightarrow$
$|f(x) - f(y)| \le r|x-y| < r \cdot \delta < \varepsilon$ (since $0 < r < 1$).

Thus $f(x)$ is continuous at each $y \in \mathbb{R}$, so $f(x)$ is continuous on $\mathbb{R}$.

Let $a_0 = y_1$, $a_1 = f(y_1)$, $a_2 = f(f(y_1))$, ...

b) We have $|a_{n+1} - a_n| = |f(a_n) - f(a_{n-1})| \le r|a_n - a_{n-1}|$
$\le \dots \le r^n |a_1 - a_0|$.

$|a_{n+m} - a_n| = |a_{n+m} - a_{n+m-1} + a_{n+m-1} - a_{n+m-2} + \dots +$

$+ a_{n+1} - a_n| \le |a_{n+m} - a_{n+m-1}| + |a_{n+m-1} - a_{n+m-2}| + \dots +$

$+ |a_{n+1} - a_n| \le r^{n+m-1}|a_1 - a_0| + r^{n+m-2}|a_1 - a_0| + \dots + r^n|a_1 - a_0|$

$\le (r^n + r^{n+1} + \dots + r^{n+m-1})|a_1 - a_0| \le r^n(1 + r + \dots + r^{m-1})|a_1 - a_0|$

$$\le r^n \cdot \sum_{k=0}^{\infty} r^k |a_1 - a_0| = \frac{r^n}{1-r}|a_1 - a_0|$$

Since $0 < r < 1$, $\lim \frac{r^n}{1-r}|a_1 - a_0| = 0$, so

given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > \ln\left(\frac{\varepsilon(1-r)}{|a_1 - a_0|}\right) / \ln r$.

Then $\forall n \ge N$ we have

$$|a_{n+m} - a_n| \le \frac{r^n}{1-r}|a_1 - a_0| \le \frac{r^N}{1-r}|a_1 - a_0| < \varepsilon.$$

c) We checked that $(a_n)$ is Cauchy, so it converges.
Let $\ell = \lim a_n$. Since
$a_{n+1} = f(a_n)$ we have $\lim a_{n+1} = \lim f(a_n)$.
Since $f$ is continuous we have
$\ell = \lim a_{n+1} = f(\lim a_n) = f(\ell)$, so $\ell$ is a fixed point.

d) Let $(b_n)$ be any other sequence defined by ②
$b_0 = x$, $b_1 = f(x)$, ..., $b_{n+1} = f(b_n)$, ...

Then $|b_{n+1} - a_{n+1}| = |f(b_n) - f(a_n)| \leq r|b_n - a_n| \leq \dots$
$\leq r^n |b_0 - a_0|$

So $a_{n+1} - r^n|b_0 - a_0| \leq b_{n+1} \leq a_{n+1} + r^n|b_0 - a_0|$

$n \to \infty \downarrow$ $\qquad\qquad$ $\downarrow n \to \infty$

$\ell$ $\qquad\qquad\qquad\qquad$ $\ell$

So $\lim b_{n+1} = \lim b_n = \ell$ by squeeze thm.

2nd proof. Suppose $m$ is any other fixed point then since $f(\ell) = \ell$ and $f(m) = m$, we have $|\ell - m| = |f(\ell) - f(m)| \leq r|\ell - m|$. Since $0 < r < 1 \Rightarrow$ $|\ell - m| = 0$ or $\ell = m$. By b) and c) $b_n \to m$ and $m$ is some fixed point of $f$, so $m = \ell$ and $b_m \to \ell$.