twk
o (Solutions)

15. In Exercise 11, we completed the square and obtained $s^2 + 2s + 10 = (s + 1)^2 + 3^2$, so

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^2 + 2s + 10}\right] &= \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2 + 3^2}\right]\\ &= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2 + 3^2}\right]\\ &= \frac{1}{3}e^{-t}\sin 3t.\end{aligned}$$

16. In Exercise 12, we completed the square and obtained $s^2 - 4s + 5 = (s - 2)^2 + 1^2$, so

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s}{s^2 - 4s + 5}\right] &= \mathcal{L}^{-1}\left[\frac{s}{(s-2)^2 + 1^2}\right]\\ &= \mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2 + 1^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s-2)^2 + 1^2}\right]\\ &= e^{2t}\cos t + e^{2t}(2\sin t) = e^{2t}(\cos t + 2\sin t).\end{aligned}$$

17. In Exercise 13, we completed the square and obtained $s^2 + s + 1 = (s + 1/2)^2 + (\sqrt{3}/2)^2$, so

$$\frac{2s+3}{s^2+s+1} = \frac{2s+3}{(s+1/2)^2 + (\sqrt{3}/2)^2}.$$

We want to put this fraction in the right form so that we can use the formulas for $\mathcal{L}[e^{at}\cos\omega t]$ and $\mathcal{L}[e^{at}\sin\omega t]$. We see that

$$\begin{aligned}\frac{2s+3}{(s+1/2)^2 + (\sqrt{3}/2)^2} &= \frac{2s+1}{(s+1/2)^2 + (\sqrt{3}/2)^2} + \frac{2}{(s+1/2)^2 + (\sqrt{3}/2)^2}\\ &= \frac{2(s+1/2)}{(s+1/2)^2 + (\sqrt{3}/2)^2} + \frac{(4/\sqrt{3})(\sqrt{3}/2)}{(s+1/2)^2 + (\sqrt{3}/2)^2}.\end{aligned}$$

So

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{2s+3}{s^2+s+1}\right] &= 2\mathcal{L}^{-1}\left[\frac{(s+1/2)}{(s+1/2)^2 + (\sqrt{3}/2)^2}\right] + \frac{4}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}/2}{(s+1/2)^2 + (\sqrt{3}/2)^2}\right]\\ &= 2e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{4}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).\end{aligned}$$

30. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 6\mathcal{L}\left[\frac{dy}{dt}\right] + 13\mathcal{L}[y] = 13\frac{e^{-4s}}{s},$$

and using the formulas for $\mathcal{L}[dy/dt]$ and $\mathcal{L}[d^2y/dt^2]$ in terms of $\mathcal{L}[y]$, we have

$$(s^2 + 6s + 13)\mathcal{L}[y] - sy(0) - y'(0) - 6y(0) = 13\frac{e^{-4s}}{s}.$$

(b) Substituting the initial conditions yields

$$(s^2 + 6s + 13)\mathcal{L}[y] - 3s - 19 = 13\frac{e^{-4s}}{s},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{3s+19}{s^2+6s+13} + \left(\frac{13}{s(s^2+6s+13)}\right)e^{-4s}.$$

Using the partial fractions decomposition

$$\frac{13}{s(s^2+6s+13)} = \frac{1}{s} - \frac{s+6}{s^2+6s+13},$$

we obtain

$$\mathcal{L}[y] = \frac{3s+19}{s^2+6s+13} + \left(\frac{1}{s} - \frac{s+6}{s^2+6s+13}\right) e^{-4s}.$$

(c) In order to compute the inverse Laplace transform, we first write

$$s^2+6s+13 = (s+3)^2+4$$

by completing the square, and then we write

$$\frac{3s+19}{s^2+6s+13} = 3\left(\frac{s+3}{(s+3)^2+4}\right) + 5\left(\frac{2}{(s+3)^2+4}\right)$$

and

$$\frac{s+6}{s^2+6s+13} = \left(\frac{s+3}{(s+3)^2+4}\right) + \frac{3}{2}\left(\frac{2}{(s+3)^2+4}\right).$$

Taking the inverse Laplace transform, we have

$$y(t) = 3e^{-3t}\cos 2t + 5e^{-3t}\sin 2t + u_4(t)\left(1 - e^{-3(t-4)}\cos 2(t-4) - \tfrac{3}{2}e^{-3(t-4)}\sin 2(t-4)\right).$$

32. **(a)** We take Laplace transform of both sides to obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 3\mathcal{L}[y] = \mathcal{L}[u_4(t)\cos(5(t-4))],$$

which is equivalent to

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 3\mathcal{L}[y] = \frac{e^{-4s}s}{s^2+25}.$$

Using the given initial conditions, we have

$$(s^2+3)\mathcal{L}[y] + 2 = \frac{e^{-4s}s}{s^2+25}.$$

(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{-2}{s^2+3} + \frac{e^{-4s}s}{(s^2+3)(s^2+25)}.$$

(c) Now to find the inverse Laplace transform, we first note that

$$\mathcal{L}^{-1}\left[\frac{-2}{s^2+3}\right] = \frac{-2}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}}{s^2+3}\right] = \frac{-2}{\sqrt{3}}\sin\sqrt{3}\,t.$$

For the second term, we first use partial fractions to write

$$\frac{s}{(s^2+3)(s^2+25)} = \frac{1}{22}\left(\frac{s}{s^2+3} - \frac{s}{s^2+25}\right).$$

Hence

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}s}{(s^2+3)(s^2+25)}\right] = \frac{1}{22}u_4(t)\left(\cos(\sqrt{3}\,(t-4)) - \cos(5(t-4))\right).$$

Combining the two results, we obtain the solution of the initial-value problem

$$y(t) = -\frac{2}{\sqrt{3}}\sin\sqrt{3}\,t + \frac{1}{22}u_4(t)\left(\cos(\sqrt{3}\,(t-4)) - \cos(5(t-4))\right).$$