Homework # 19 (Solutions)

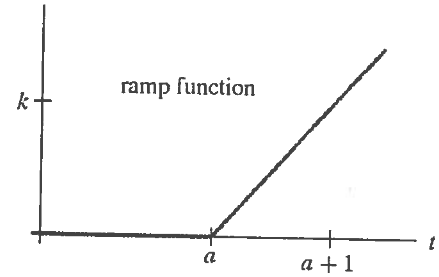
2. **(a)** We have $r_a(t) = u_a(t)y(t-a)$, where $y(t) = kt$. Now

$$\mathcal{L}[y(t)] = k\mathcal{L}[t] = \frac{k}{s^2},$$

so using the rules of Laplace transform,

$$\mathcal{L}[r_a(t)] = \mathcal{L}[u_a(t)y(t-a)] = \frac{k}{s^2}e^{-as}.$$

(b)



6. Using partial fractions, we write

$$\frac{4}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}.$$

Hence, we must have $As + 3A + Bs = 4$ which can be written as $(A+B)s + 3A = 4$. So, $A + B = 0$, and $3A = 4$. This gives us $A = 4/3$ and $B = -4/3$, so

$$\frac{4}{s(s+3)} = \frac{4/3}{s} - \frac{4/3}{s+3}.$$

Applying the rules

$$\mathcal{L}[u_2(t)] = \frac{e^{-2s}}{s}$$

and

$$\mathcal{L}[u_2(t)e^{-3(t-2)}] = \frac{e^{-2s}}{s+3},$$

the desired function is

$$y(t) = u_2(t)\left(\frac{4}{3} - \frac{4e^{-3(t-2)}}{3}\right)$$

or

$$y(t) = \frac{4}{3}u_2(t)\left(1 - e^{-3(t-2)}\right).$$

7. Using partial fractions, we get

$$\frac{14}{(3s+2)(s-4)} = \frac{A}{3s+2} + \frac{B}{s-4}.$$

Hence, we must have $As - 4A + 3Bs + 2B = 14$, which can be written as

$$(A + 3B)s + (-4A + 2B) = 14.$$

Therefore, $A + 3B = 0$, and $-4A + 2B = 14$. Solving for A and B yields $A = -3$ and $B = 1$, so

$$\frac{14}{(3s+2)(s-4)} = \frac{1}{s-4} - \frac{3}{3s+2} = \frac{1}{s-4} - \frac{1}{s+2/3}.$$

Applying the rules

$$\mathcal{L}[u_1(t)e^{4(t-1)}] = \frac{e^{-s}}{s-4}$$

and

$$\mathcal{L}[u_1(t)e^{-\frac{2}{3}(t-1)}] = \frac{e^{-s}}{s+2/3},$$

the desired function is

$$y(t) = u_1(t)\left(e^{4(t-1)} - e^{-\frac{2}{3}(t-1)}\right).$$

11. Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \mathcal{L}[-y] + \mathcal{L}[u_2(t)e^{-2(t-2)}],$$

which is equivalent to

$$s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + \frac{e^{-2s}}{s+2}$$

(using linearity of the Laplace transform and the formula

If $\mathcal{L}[f] = F(s)$ then $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$

where $f(t) = e^{-2t}$ and $a = 2$.)

Substituting the initial condition yields

$$s\mathcal{L}[y] - 1 = -\mathcal{L}[y] + \frac{e^{-2s}}{s+2}$$

so that

$$\mathcal{L}[y] = \frac{1}{s+1} + \frac{e^{-2s}}{(s+1)(s+2)}.$$

By partial fractions, we know that

$$\frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)(s+2)},$$

so we have

$$\frac{e^{-2s}}{(s+1)(s+2)} = e^{-2s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) = \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s+2}.$$

Taking the inverse of the Laplace transform yields

$$\begin{aligned} y(t) &= e^{-t} + u_2(t)e^{-(t-2)} - u_2(t)e^{-2(t-2)} \\ &= e^{-t} + u_2(t)\left(e^{-(t-2)} - e^{-2(t-2)}\right). \end{aligned}$$

To check our answer, we compute

$$\frac{dy}{dt} = -e^{-t} + \frac{du_2}{dt}\left(e^{-(t-2)} - e^{-2(t-2)}\right) + u_2(t)\left(-e^{-(t-2)} + 2e^{-2(t-2)}\right),$$

and since $du_2/dt = 0$ except at $t = 2$ (where it is undefined),

$$\begin{aligned} \frac{dy}{dt} + y &= -e^{-t} + u_2(t)\left(-e^{-(t-2)} + 2e^{-2(t-2)}\right) + e^{-t} + u_2(t)\left(e^{-(t-2)} - e^{-2(t-2)}\right) \\ &= u_2(t)e^{-2(t-2)}. \end{aligned}$$

Hence, our $y(t)$ satisfies the differential equation except when $t = 2$. (We cannot expect $y(t)$ to satisfy the differential equation at $t = 2$ because the differential equation is not continuous there.) Note that $y(t)$ also satisfies the initial condition $y(0) = 1$.

13. Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{dy}{dt}\right] = -\mathcal{L}[y] + \mathcal{L}[u_1(t)(t-1)],$$

which is equivalent to

$$s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + \frac{e^{-s}}{s^2}.$$

Substituting the initial condition yields

$$s\mathcal{L}[y] - 2 = -\mathcal{L}[y] + \frac{e^{-s}}{s^2}$$

so that

$$\mathcal{L}[y] = \frac{e^{-s}}{s^2(s+1)} + \frac{2}{s+1}.$$

Using the technique of partial fractions, we write

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}.$$

Putting the right-hand side over a common denominator gives us $As(s+1) + B(s+1) + Cs^2 = 1$ which can be written as $(A+C)s^2 + (A+B)s + B = 1$. So $A + C = 0$, $A + B = 0$, and $B = 1$. Thus $A = -1$ and $C = 1$, and

$$\frac{1}{s^2(s+1)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}.$$

Taking the inverse of the Laplace transform gives us

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2(s+1)}\right] + \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] \\
&= -\mathcal{L}^{-1}\left[\frac{e^{-s}}{s}\right] + \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{e^{-s}}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] \\
&= -u_1(t) + u_1(t)\,(t-1) + u_1(t)e^{-(t-1)} + 2e^{-t} \\
&= u_1(t)\left((t-2) + e^{-(t-1)}\right) + 2e^{-t}.
\end{aligned}$$

To check our answer, we compute

$$\frac{dy}{dt} = \frac{du_1}{dt}\left((t-2) + e^{-(t-1)}\right) + u_1(t)\left(1 - e^{-(t-1)}\right) - 2e^{-t},$$

and since $du_1/dt = 0$ except at $t = 1$ (where it is undefined),

$$\begin{aligned}
\frac{dy}{dt} + y &= u_1(t)\left(1 - e^{-(t-1)}\right) - 2e^{-t} + u_1(t)\left((t-2) + e^{-(t-1)}\right) + 2e^{-t} \\
&= u_1(t) + u_1(t)\,(t-2) \\
&= u_1(t)\,(t-1)\,.
\end{aligned}$$

Hence, our $y(t)$ satisfies the differential equation except when $t = 1$. (We cannot expect $y(t)$ to satisfy the differential equation at $t = 1$ because the differential equation is not continuous there.) Note that $y(t)$ also satisfies the initial condition $y(0) = 2$.