Homework # 18 (Solutions)

2. We have

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt = \lim_{b\to\infty}\int_0^b te^{-st}\,dt.$$

To evaluate the integral we use integration by parts with $u = t$ and $dv = e^{-st}dt$. Then $du = dt$ and $v = -e^{-st}/s$. Thus

$$\begin{aligned}\lim_{b\to\infty}\int_0^b te^{-st}\,dt &= \lim_{b\to\infty}\left(-\frac{te^{-st}}{s}\Big|_0^b - \int_0^b -\frac{e^{-st}}{s}\,dt\right)\\ &= \lim_{b\to\infty}\left(-\frac{be^{-sb}}{s} - \frac{e^{-st}}{s^2}\Big|_0^b\right)\\ &= \lim_{b\to\infty}\left(-\frac{be^{-sb}}{s} - \frac{e^{-sb}}{s^2} + \frac{e^0}{s^2}\right)\\ &= \frac{1}{s^2}\end{aligned}$$

since

$$\lim_{b\to\infty} -\frac{be^{-sb}}{s} = \lim_{b\to\infty}\frac{-b}{se^{sb}} = \lim_{b\to\infty}\frac{-1}{s^2e^{sb}} = 0$$

by L'Hôpital's Rule if $s > 0$.

12. Using the method of partial fractions, we write

$$\frac{5}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

Putting the right-hand side over a common denominator gives $A(s-2) + B(s-1) = 5$, which can be written as $(A+B)s + (-2A-B) = 5$. Thus, $A+B=0$, and $-2A-B=5$. This gives $A=-5$ and $B=5$, so

$$\mathcal{L}^{-1}\left[\frac{5}{(s-1)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{5}{s-2} - \frac{5}{s-1}\right].$$

Thus,

$$\mathcal{L}^{-1}\left[\frac{5}{(s-1)(s-2)}\right] = 5e^{2t} - 5e^t.$$

14. Using the method of partial fractions,

$$\frac{2s^2+3s-2}{s(s+1)(s-2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-2}.$$

Putting the right-hand side over a common denominator gives

$$A(s+1)(s-2) + Bs(s-2) + Cs(s+1) = 2s^2+3s-2,$$

which can be written as $(A+B+C)s^2 + (-A-2B+C)s - 2A = 2s^2+3s-2$. So, $A+B+C=2$, $-A-2B+C=3$, and $-2A=-2$. Thus $A=1$, $B=-1$, and $C=2$, and

$$\mathcal{L}^{-1}\left[\frac{2s^2+3s-2}{s(s+1)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s-2} - \frac{1}{s+1} + \frac{1}{s}\right].$$

Hence,

$$\mathcal{L}^{-1}\left[\frac{2s^2+3s-2}{s(s+1)(s-2)}\right] = 2e^{2t} - e^{-t} + 1.$$

16. **(a)** Taking Laplace transforms of both sides of the equation and simplifying gives

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 5\mathcal{L}[y] = \mathcal{L}[e^{-t}]$$

so

$$s\mathcal{L}[y] - y(0) + 5\mathcal{L}[y] = \frac{1}{s+1}$$

and $y(0) = 2$ gives

$$s\mathcal{L}[y] - 2 + 5\mathcal{L}[y] = \frac{1}{s+1}.$$

(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{2}{s+5} + \frac{1}{(s+5)(s+1)} = \frac{2s+3}{(s+5)(s+1)}.$$

(c) Using the method of partial fractions,

$$\frac{2s+3}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}.$$

Putting the right-hand side over a common denominator gives $A(s+1) + B(s+5) = 2s+3$, which can be written as $(A+B)s + (A+5B) = 2s+3$. So $A+B = 2$, and $A+5B = 3$. Hence, $A = 7/4$ and $B = 1/4$ and we have

$$\mathcal{L}[y] = \frac{7/4}{s+5} + \frac{1/4}{s+1}.$$

Therefore,

$$y(t) = \frac{7}{4}e^{-5t} + \frac{1}{4}e^{-t}.$$

(d) To check, compute

$$\frac{dy}{dt} + 5y = -\frac{35}{4}e^{-5t} - \frac{1}{4}e^{-t} + 5\left(\frac{7}{4}e^{-5t} + \frac{1}{4}e^{-t}\right) = e^{-t},$$

and $y(0) = 7/4 + 1/4 = 2$.

22. **(a)** Putting the equation in the form

$$\frac{dy}{dt} - 2y = t,$$

taking Laplace transforms of both sides of the equation and simplifying gives

$$\mathcal{L}\left[\frac{dy}{dt}\right] - 2\mathcal{L}[y] = \mathcal{L}[t].$$

Using the formulas

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0),$$

and

$$\mathcal{L}[t^n] = n!/s^{n+1},$$

we have

$$s\mathcal{L}[y] - y(0) - 2\mathcal{L}[y] = \frac{1}{s^2}.$$

The initial condition $y(0) = 0$ gives

$$s\mathcal{L}[y] - 2\mathcal{L}[y] = \frac{1}{s^2}.$$

(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{1}{s^2(s-2)}.$$

(c) Using partial fractions, we seek constants A, B, and C so that

$$\frac{1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}.$$

Putting the right-hand side over a common denominator gives $As(s-2) + B(s-2) + Cs^2 = 1$, which can be written as $(A+C)s^2 + (-2A+B)s - 2B = 1$. This gives us $A + C = 0$, $-2A + B = 0$, and $-2B = 1$. Hence, $A = -1/4$, $B = -1/2$, and $C = 1/4$, and we get

$$\mathcal{L}[y] = \frac{1/4}{s-2} - \frac{1/2}{s^2} - \frac{1/4}{s}.$$

So,

$$y(t) = \frac{1}{4}e^{2t} - \frac{t}{2} - \frac{1}{4}.$$

(d) To check, we compute

$$\frac{dy}{dt} - 2y = \frac{1}{2}e^{2t} - \frac{1}{2} - 2\left(\frac{1}{4}e^{2t} - \frac{t}{2} - \frac{1}{4}\right) = t,$$

and $y(0) = 1/4 - 1/4 = 0$, so our solution satisfies the initial-value problem.