Homework #17 (Solutions)

1. We plug in: $(y_p + k_1 y_1 + k_2 y_2)'' + p(y_p + k_1 y_1 + k_2 y_2)'$
$+ q(y_p + k_1 y_1 + k_2 y_2) = [y_p'' + p y_p' + q y_p] +$
$+ [(k_1 y_1 + k_2 y_2)'' + p(k_1 y_1 + k_2 y_2)' + q(k_1 y_1 + k_2 y_2)]$
$= f + 0 = f.$

Thus $y_p + k_1 y_1 + k_2 y_2$ is a solution of $y'' + py' + qy = f$.

2 I) a) $y'' + 4y' + 5y = t + e^{-2t}$

- Seek particular solution as $y_p(t) = At + B + Ce^{-2t}$.

We plug in: $4Ce^{-2t} + 4(A - 2Ce^{-2t}) + 5(At + B + Ce^{-2t})$
$= Ce^{-2t} + 5At + (4A + 5B) = t + e^{-2t}$

$\begin{cases} C = 1 \\ 5A = 1 \\ 4A + 5B = 0 \end{cases}$ $\quad A = 1/5$, $B = -4/25$ $\quad y_p(t) = \frac{1}{5}t - \frac{4}{25} + e^{-2t}$

- Now we find the general solution of $y'' + 4y' + 5y = 0$.

$p(\lambda) = \lambda^2 + 4\lambda + 5 = (\lambda + 2)^2 + 1 = 0$, $\lambda_{1,2} = -2 \pm i$.

Gen. solution is $e^{-2t}(k_1 \cos t + k_2 \sin t)$.

- Gen. solution of $y'' + 4y' + 5y = t + e^{-2t}$ is

$$\boxed{y(t) = 1/5\, t - \frac{4}{25} + e^{-2t} + e^{-2t}(k_1 \cos t + k_2 \sin t)}$$

b) $y(0) = -4/25 + 1 + k_1 = 0$, $k_1 = -1 + 4/25 = -\frac{21}{25}$

$y'(0) = 1/5 - 2 - 2k_1 + k_2 = 0$

$k_2 = -1/5 + 2 - \frac{42}{25} = \frac{-5 + 50 - 42}{25} = \frac{3}{25}$

$y(t) = 1/5\, t - 4/25 + e^{-2t} + e^{-2t}\left(-\frac{21}{25}\cos(2t) + \frac{3}{25}\sin(2t)\right)$

c) Transient: $e^{-2t} + e^{-2t}\left(-\frac{21}{25}\cos(2t) + \frac{3}{25}\sin(2t)\right)$

Steady-state: $1/5\, t - 4/25$.

Forced response: $1/5\, t - 4/25 + e^{-2t}$.

Free response: $e^{-2t}\left(-\frac{21}{25}\cos(2t) + \frac{3}{25}\sin(2t)\right)$.

II) a) • $y_p(t) = A\cos(2t) + B\sin(2t)$. Plug in:

$$\left(-4A\cos(2t) - 4B\sin(2t)\right) + 4\left(-2A\sin(2t) + 2B\cos(2t)\right)$$
$$+3\left(A\cos(2t) + B\sin(2t)\right)$$

$$= (-4A + 8B + 3A)\cos(2t) + (-4B - 8A + 3B)\sin(2t) = \cos(2t)$$

$$\begin{cases} -A + 8B = 1 \\ -B - 8A = 0 \end{cases}, \quad B = -8A, \quad -A - 64A = 1, \quad A = -\frac{1}{65}, \quad B = 8/65$$

$$y_p(t) = -\frac{1}{65}\cos(2t) + \frac{8}{65}\sin(2t).$$

• Solving $y'' + 4y' + 3y = 0$: $\lambda^2 + 4\lambda + 3 = 0$

$\lambda_1 = -1$, $\lambda_2 = -3$. Gen. sol. is $k_1 e^{-t} + k_2 e^{-3t}$.

• General solution of $y'' + 4y' + 3y = \cos(2t)$ is

$$\boxed{y(t) = -1/65\cos(2t) + \frac{8}{65}\sin(2t) + k_1 e^{-t} + k_2 e^{-3t}}$$

b) $$\left.\begin{aligned} y(0) &= -1/65 + k_1 + k_2 = 0 \\ &\frac{16}{65} - k_1 - 3k_2 = 0 \end{aligned}\right\} \Rightarrow \frac{15}{65} - 2k_2 = 0$$

$$k_2 = \frac{15}{130} = 3/26$$

$$k_1 = \frac{16}{65} - 3k_2 = \frac{16}{65} - \frac{45}{130} = -\frac{13}{130} = -1/10$$

$$\boxed{y(t) = -\frac{1}{65}\cos(2t) + \frac{8}{65}\sin(2t) - \frac{1}{10}e^{-t} + 3/26\, e^{-3t}}$$

c) Steady state and forced response:

$$-\frac{1}{65}\cos(2t) + \frac{8}{65}\sin(2t)$$

Transient and free response:

$$-\frac{1}{10}e^{-t} + \frac{3}{26}e^{-3t}.$$

III. a) $y_p(t) = (At+B)e^{-t}$. Plug in:

$$(-2Ae^{-t} + (At+B)e^{-t}) + 9(At+B)e^{-t} = te^{-t}$$

$$(A+9A)te^{-t} + (B+9B-2A)e^{-t} = te^{-t}$$

$$\begin{cases} 10A = 1 & , A = 1/10 \\ 10B - 2A = 0 & , B = \frac{1}{5}A = \frac{1}{50} \end{cases} \quad y_p(t) = \left(\frac{1}{10}t + \frac{1}{50}\right)e^{-t}$$

• Solving $y'' + 9y = 0$; $p(\lambda) = \lambda^2 + 9$, $\lambda_{1,2} = \pm 3i$.

Gen. sol. is $k_1\cos(3t) + k_2\sin(3t)$

• Gen. solution of $y'' + 9y = te^{-t}$ is

$$\boxed{y(t) = \left(\frac{1}{10}t + \frac{1}{50}\right)e^{-t} + k_1\cos(3t) + k_2\sin(3t)}$$

b) $y(0) = 1/50 + k_1 = 0$, $k_1 = -1/50$

$y'(0) = \frac{1}{10} - \frac{1}{50} + 3k_2 = 0$, $k_2 = \frac{-4}{150}$

$$\boxed{y(t) = \underbrace{\left(\frac{1}{10}t + \frac{1}{50}\right)e^{-t}}_{\text{forced response, transient}} \underbrace{- \frac{1}{50}\cos(3t) - \frac{4}{150}\sin(3t)}_{\text{free response, steady state}}}$$

3. a) We plug in:

y_1: $(e^{-2t})'' + 4(e^{-2t})' + 4e^{-2t} = 4e^{-2t} - 8e^{-2t} + 4e^{-2t} = 0$

y_2: $(te^{-2t})'' + 4(te^{-2t})' + 4(te^{-2t})$

$= -4e^{-2t} + 4te^{-2t} + 4e^{-2t} - 8te^{-2t} + 4te^{-2t} = 0$

Remark: Thus, the general solution of $y'' + 4y' + 4y = 0$ is $k_1 e^{-2t} + k_2 t e^{-2t}$.

b) We try $y_p(t) = At^2e^{-2t}$. We plug in:

$(At^2e^{-2t})'' + 4(At^2e^{-2t})' + 4At^2e^{-2t} =$

$2Ae^{-2t} - \cancel{8Ate^{-2t}} + \cancel{4At^2e^{-2t}} + \cancel{8Ate^{-2t}} - \cancel{8At^2e^{-2t}}$

$+ \cancel{4At^2e^{-2t}} = 2Ae^{-2t} = e^{-2t} \Rightarrow 2A = 1, \ A = 1/2.$

Thus, $\boxed{y_p(t) = \frac{1}{2}t^2e^{-2t}}$