Homework #16 (Solutions)

1. We plug $k_1y_1+k_2y_2$ into $y''+py'+qy$:

$$(k_1y_1+k_2y_2)''+p(k_1y_1+k_2y_2)'+q(k_1y_1+k_2y_2)$$
$$=k_1(y_1''+py_1'+qy_1)+k_2(y_2''+py_2'+qy_2)=$$
$$=k_1\cdot 0+k_2\cdot 0=0$$ (Both $y_1''+py_1'+qy_1=0$ and $y_2''+py_2'+qy_2=0$ since y_1 and y_2 are solutions).

We conclude that $k_1y_1+k_2y_2$ is also a solution since it satisfies diff. equation

2. Since it is given that $y_{re}+iy_{im}$ is a solution of $y''+py'+qy=0$, we have

$$(y_{re}+iy_{im})''+p(y_{re}+iy_{im})'+q(y_{re}+iy_{im})=$$
$$=\underbrace{(y_{re}''+py_{re}'+qy_{re})}+i\underbrace{(y_{im}''+py_{im}'+qy_{im})}=0$$

Since $a+bi=0+i0$ for real $a,b \Rightarrow a=0$ and $b=0$, we have $y_{re}''+py_{re}'+qy_{re}=0$ and $y_{im}''+py_{im}'+qy_{im}=0$

$\Rightarrow$ y_{re} and y_{im} are two solutions of $y''+py'+qy=0$.

3. I: $y''+2y'+10y=0$

a) $D=2^2-4\cdot 10=-36<0 \Rightarrow$ underdamped.

b) $p(\lambda)=\lambda^2+2\lambda+10=0$

$$\lambda_{1,2}=\frac{-2\pm\sqrt{4-40}}{2}=-1\pm 3i$$

Complex solution is $y_1(t)=e^{(-1+3i)t}$

By Euler's formula $e^{a+bi} = e^a(\cos b + i \sin b)$,
we have

$$y_1(t) = e^{-t}(\cos(3t) + i\sin(3t)) = \underbrace{e^{-t}\cos(3t)}_{y_{re}} + i\underbrace{e^{-3t}\sin(3t)}_{y_{im}}$$

General solution is

$$\boxed{y(t) = K_1 y_{re}(t) + K_2 y_{im}(t) = e^{-t}(K_1\cos(3t) + K_2\sin(3t))}$$

Natural period is $T = \frac{2\pi}{\beta} = \frac{2\pi}{3}$, natural frequency
$w = \frac{1}{T} = \beta/2\pi = \frac{3}{2\pi}$.

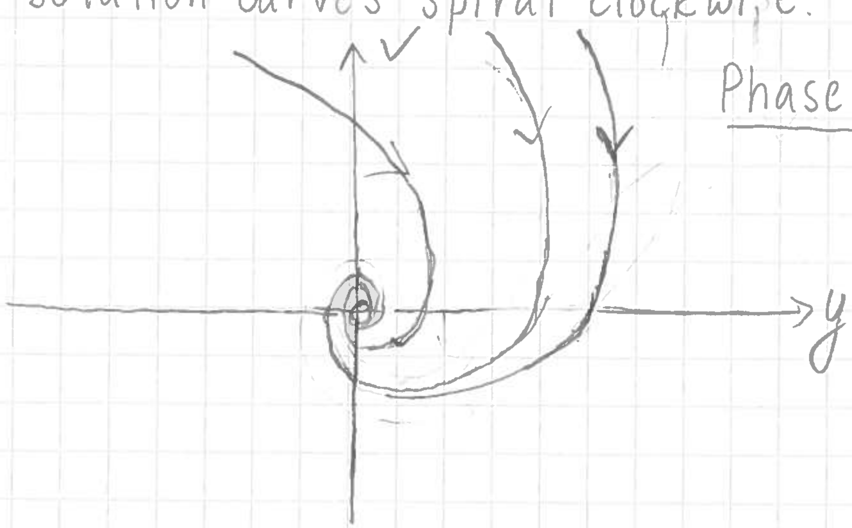
c) $y(0) = \boxed{K_1 = 4}$

$y'(0) = -K + 3K_2 = -2$, $\boxed{K_2 = 2/3}$

Answer: $y(t) = e^{-t}(4\cos(3t) + \frac{2}{3}\sin(3t))$

d) $\begin{cases} dy/dt = v \\ dv/dt = -10y - 2v \end{cases}$ $\lambda_1 = -1 + 3i = \alpha + i\beta$,
$\alpha = -1 \Rightarrow$ phase portrait is a spiral sink.

Vector field is
$V(y, v) = (v, -10y - 2v)$, $V(1,0) = (0, -10) \Rightarrow$
solution curves spiral clockwise.



Phase portrait.

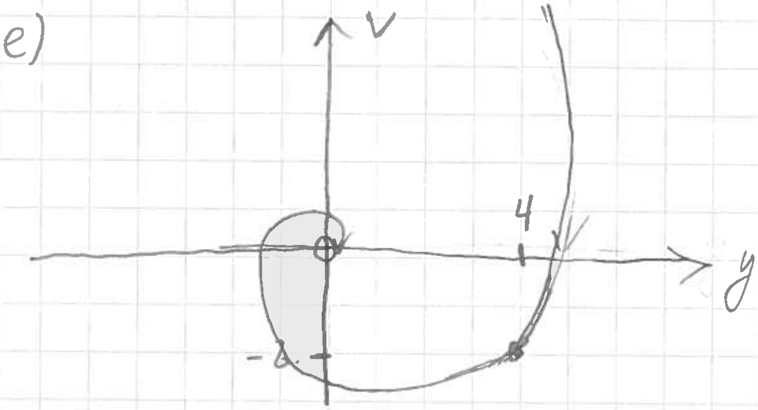
General solution :

$$Y(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}(k_1 \cos(3t) + k_2 \sin(3t)) \\ -e^{-t}(k_1 \cos(3t) + k_2 \sin(3t)) + e^{-t}(-3k_1 \sin(3t) + 3k_2 \cos(3t)) \end{pmatrix}$$

$$= e^{-t}\left(k_1 \begin{pmatrix} \cos(3t) \\ k_1 \cos(3t) - 3\sin(3t) \end{pmatrix} + k_2 \begin{pmatrix} \sin(3t) \\ \sin(3t) + 3\cos(3t) \end{pmatrix} \right)$$

e)

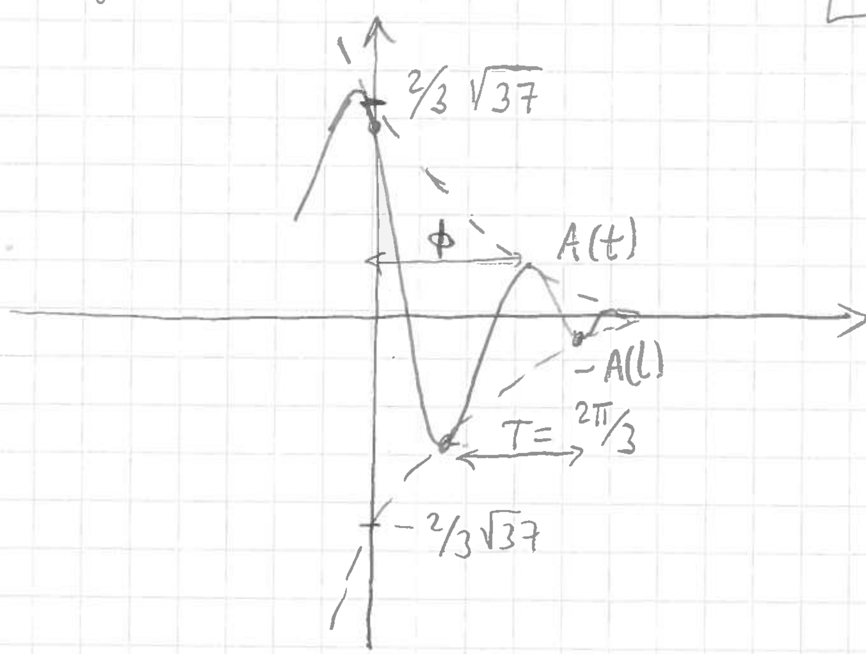


f) $y(t) = e^{-t}(4\cos(3t) + \frac{2}{3}\sin(3t)) = e^{-t}(K\cos(3t - \phi))$

$K = \sqrt{k_1^2 + k_2^2} = \sqrt{16 + 4/9} = \frac{2}{3}\sqrt{37}$

$y(t) = A(t)\cos(3t - \phi)$, $\boxed{A(t) = \frac{2}{3}\sqrt{37}\, e^{-t}}$

variable amplitude.



II. a) $y'' + 6y' + 5y = 0$. $D = 36 - 20 = 16 > 0$

$\Rightarrow$ overdamped.

b) $p(\lambda) = \lambda^2 + 6\lambda + 5 = (\lambda+5)(\lambda+1)$ $\lambda_1 = -5$, $\lambda_2 = -1$.

$$\boxed{y(t) = k_1 e^{-5t} + k_2 e^{-t}}$$

c) $y(0) = k_1 + k_2 = 4 \Rightarrow -4k_1 = 2$, $k_1 = -\frac{1}{2}$

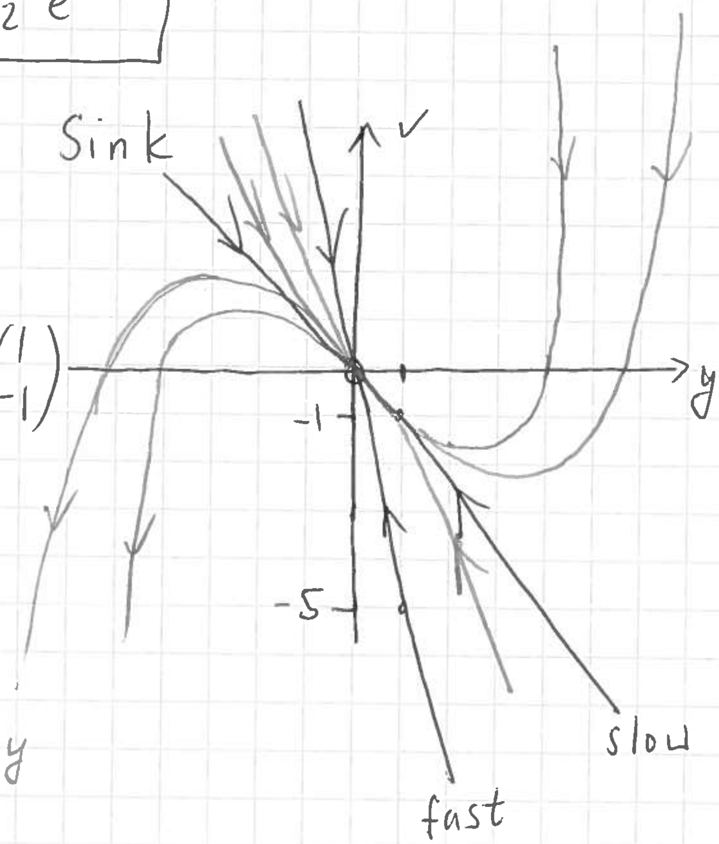
$y'(0) = -5k_1 - k_2 = -2 \Rightarrow k_2 = \frac{9}{2}$

$$\boxed{y(t) = -\tfrac{1}{2} e^{-5t} + \tfrac{9}{2} e^{-t}}$$

d) $\begin{cases} dy/dt = v \\ dv/dt = -5y - 6v \end{cases}$

General solution is

$$Y(t) = \begin{pmatrix} y \\ v \end{pmatrix} = k_1 e^{-5t}\begin{pmatrix} 1 \\ -5 \end{pmatrix} + k_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



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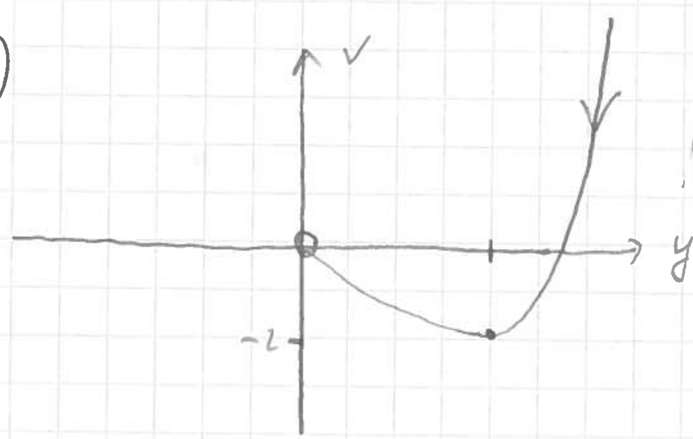

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