Solutions to homework 15

Sunday, November 11, 2018 7:55 PM

4. **(a)** The characteristic equation is

$$(2-\lambda)(6-\lambda)+8=\lambda^2-8\lambda+20,$$

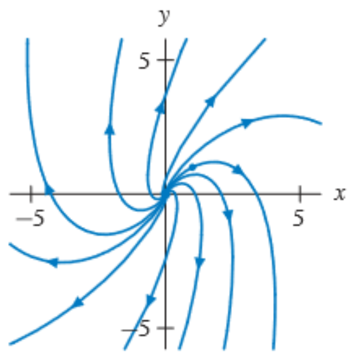
and the eigenvalues are $\lambda = 4 \pm 2i$.

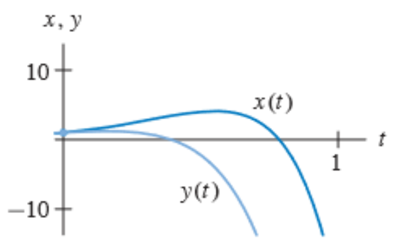
(b) Since the real part of the eigenvalues is positive, the origin is a spiral source.

(c) Since $\lambda = 4 \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

(d) At the point $(1, 0)$, the tangent vector is $(2, -4)$. Therefore, the solution curves spiral around the origin in a clockwise fashion.

(e) Since $d\mathbf{Y}/dt = (4, 2)$ at $\mathbf{Y}_0 = (1, 1)$, both $x(t)$ and $y(t)$ increase initially. The distance between successive zeros is π, and the amplitudes of both $x(t)$ and $y(t)$ are increasing.





6. **(a)** The characteristic polynomial is

$$(-\lambda)(-1-\lambda)+4=\lambda^2+\lambda+4,$$

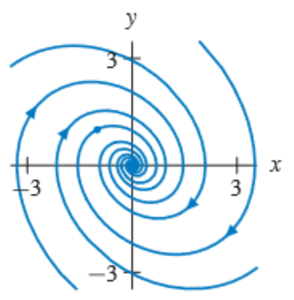
so the eigenvalues are $\lambda = (-1 \pm i\sqrt{15})/2$.

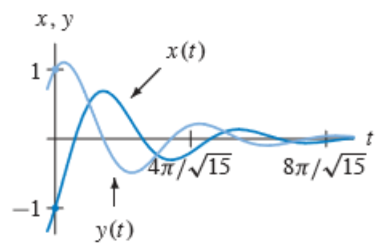
(b) The eigenvalues are complex and the real part is negative, so the origin is a spiral sink.

(c) The natural period is $2\pi/(\sqrt{15}/2) = 4\pi/\sqrt{15}$. The natural frequency is $\sqrt{15}/(4\pi)$.

(d) The vector field at $(1, 0)$ is $(0, -2)$. Hence, solution curves spiral about the origin in a clockwise fashion.

(e) From the phase plane, we see that both $x(t)$ and $y(t)$ are initially increasing. However, $y(t)$ quickly reaches a local maximum. Although both functions oscillate, each successive oscillation has a smaller amplitude.





10. **(a)** According to Exercise 4, the eigenvalues are $\lambda = 4 \pm 2i$. The eigenvectors (x, y) associated to the eigenvalue $4+2i$ must satisfy the equation $y = (1+i)x$. Hence, using the eigenvector $(1, 1+i)$, we obtain the complex-valued solution

$$\mathbf{Y}(t) = e^{(4+2i)t}\begin{pmatrix}1\\1+i\end{pmatrix} = e^{4t}\begin{pmatrix}\cos 2t\\ \cos 2t - \sin 2t\end{pmatrix} + ie^{4t}\begin{pmatrix}\sin 2t\\ \cos 2t + \sin 2t\end{pmatrix}.$$

From the real and imaginary parts of $\mathbf{Y}(t)$, we obtain the general solution

$$\mathbf{Y}(t) = k_1e^{4t}\begin{pmatrix}\cos 2t\\ \cos 2t - \sin 2t\end{pmatrix} + k_2e^{4t}\begin{pmatrix}\sin 2t\\ \cos 2t + \sin 2t\end{pmatrix}.$$

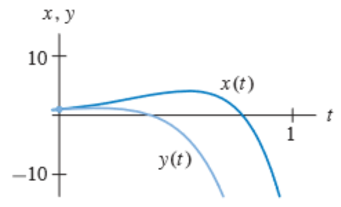
(b) Using the initial condition, we have

$$k_1\begin{pmatrix}1\\1\end{pmatrix} + k_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix},$$

and thus $k_1 = 1$ and $k_2 = 0$. The desired solution is

$$\mathbf{Y}(t) = e^{4t}\begin{pmatrix}\cos 2t\\ \cos 2t - \sin 2t\end{pmatrix}.$$

(c)



12. **(a)** The eigenvalues are the roots of the characteristic polynomial

$$(-\lambda)(-1-\lambda)+4=\lambda^2+\lambda+4.$$

So $\lambda = (-1 \pm i\sqrt{15})/2$. The eigenvectors (x, y) associated to the eigenvalue $\lambda = (-1+i\sqrt{15})/2$ must satisfy the equation $4y = (-1+i\sqrt{15})x$. Hence, $(4, -1+i\sqrt{15})$ is an eigenvector for this eigenvalue, and we have the complex-valued solution

$$\mathbf{Y}(t) = e^{(-1+i\sqrt{15})t/2}\begin{pmatrix}4\\ -1+i\sqrt{15}\end{pmatrix}$$

$$= e^{-t/2}\left(\cos\left(\tfrac{\sqrt{15}}{2}t\right) + i\sin\left(\tfrac{\sqrt{15}}{2}t\right)\right)\begin{pmatrix}4\\ -1+i\sqrt{15}\end{pmatrix}$$

$$= e^{-t/2}\begin{pmatrix}4\cos\left(\frac{\sqrt{15}}{2}t\right)\\ -\cos\left(\frac{\sqrt{15}}{2}t\right) - \sqrt{15}\sin\left(\frac{\sqrt{15}}{2}t\right)\end{pmatrix} +$$

$$ie^{-t/2}\begin{pmatrix}4\sin\left(\frac{\sqrt{15}}{2}t\right)\\ -\sin\left(\frac{\sqrt{15}}{2}t\right) + \sqrt{15}\cos\left(\frac{\sqrt{15}}{2}t\right)\end{pmatrix}.$$

By taking real and imaginary parts

$$\mathbf{Y}_{\text{re}}(t) = e^{-t/2}\begin{pmatrix}4\cos\left(\frac{\sqrt{15}}{2}t\right)\\ -\cos\left(\frac{\sqrt{15}}{2}t\right) - \sqrt{15}\sin\left(\frac{\sqrt{15}}{2}t\right)\end{pmatrix}$$

and

$$\mathbf{Y}_{\text{im}}(t) = e^{-t/2}\begin{pmatrix}4\sin\left(\frac{\sqrt{15}}{2}t\right)\\ -\sin\left(\frac{\sqrt{15}}{2}t\right) + \sqrt{15}\cos\left(\frac{\sqrt{15}}{2}t\right)\end{pmatrix},$$

we form the general solution $k_1\mathbf{Y}_{\text{re}}(t) + k_2\mathbf{Y}_{\text{im}}(t)$.

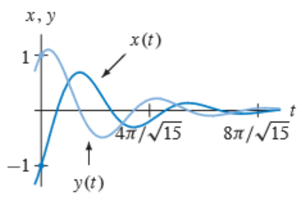
(b) To find the particular solution with the initial condition $(-1, 1)$ we set $t = 0$ in the general solution and solve for k_1 and k_2. We get

$$\begin{cases} 4k_1 = -1\\ -k_1 + \sqrt{15}k_2 = 1, \end{cases}$$

which yields $k_1 = -1/4$ and $k_2 = 3/(\sqrt{15}4) = \sqrt{15}/20$. The desired solution is

$$\mathbf{Y}(t) = e^{-t/2}\begin{pmatrix}-\cos\left(\frac{\sqrt{15}}{2}t\right) + \frac{\sqrt{15}}{5}\sin\left(\frac{\sqrt{15}}{2}t\right)\\ \cos\left(\frac{\sqrt{15}}{2}t\right) + \frac{\sqrt{15}}{5}\sin\left(\frac{\sqrt{15}}{2}t\right)\end{pmatrix}.$$

(c)



22. Consider the point in the plane determined by the coordinates (k_1, k_2), and let ϕ be an angle such that $K\cos\phi = k_1$ and $K\sin\phi = k_2$. (Such an angle exists since $(K\cos\phi, K\sin\phi)$ parameterizes the circle through (k_1, k_2) centered at the origin. In fact, there are infinitely many such ϕ, all differing by integer multiples of 2π.) Then

$$\begin{aligned} x(t) &= k_1\cos\beta t + k_2\sin\beta t\\ &= K\cos\phi\cos\beta t + K\sin\phi\sin\beta t\\ &= K\cos(\beta t - \phi). \end{aligned}$$

The last equality comes from the trigonometric identity for the cosine of the difference of two angles.