Homework #14.

1) a) For any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$\lambda_1 + \lambda_2 = a + d$ and $\lambda_1 \lambda_2 = ad - bc$

For our matrix $\lambda_1 + \lambda_2 = 1 + 3 = 4$, so

$\boxed{\lambda_2 = 4 - \lambda_1 = 3 - \sqrt{3}}$.

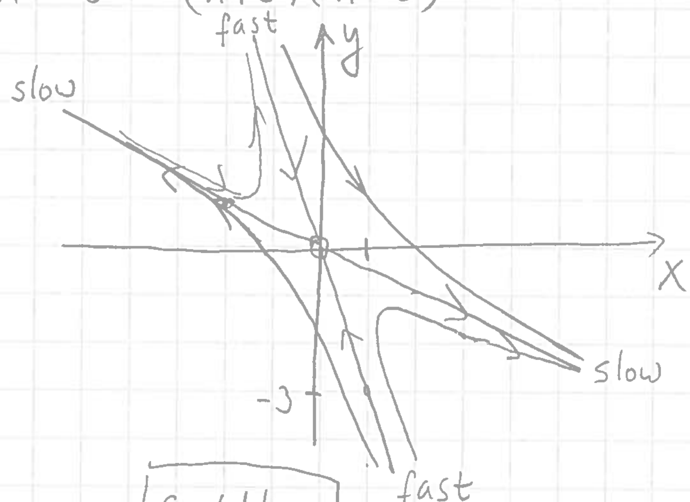
b) $\lambda_1 \cdot \lambda_2 = 1 \cdot 3 - bc$

$\underbrace{(1+\sqrt{3})(3-\sqrt{3})}_{= 2\sqrt{3}} = 3 - bc \Rightarrow \boxed{bc = 3 - 2\sqrt{3}}$

2) a) $A = \begin{pmatrix} 3 & 2 \\ -3 & -4 \end{pmatrix}$, $p(\lambda) = \det \begin{pmatrix} 3-\lambda & 2 \\ -3 & -4-\lambda \end{pmatrix}$

$= (3-\lambda)(-4-\lambda) + 6 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$

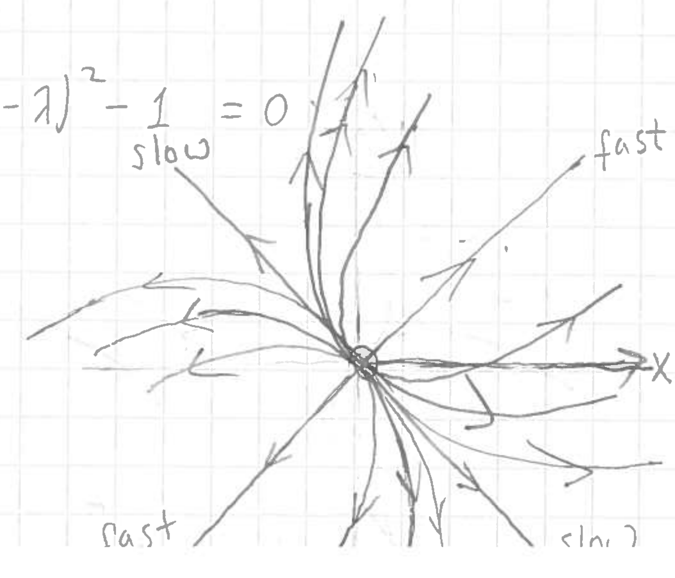
$\lambda_1 = -3$,	$\lambda_2 = 2$
$6u + 2v = 0$	$u + 2v = 0$
$v = -3u$	$u = -2v$
$V_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$	$V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
fast	slow



$\boxed{\text{Saddle}}$

b) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, $p(\lambda) = (3-\lambda)^2 - 1 = 0$

$\boxed{\lambda_1 = 2,\ \lambda_2 = 4}$ $\boxed{\text{Source!}}$



$\lambda_1 = 2$	$\lambda_2 = 4$
$u + v = 0$	$-u + v = 0$
$V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
slow	fast

c) Corresponding matrix is

$A=\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ for the system $\begin{cases} dy/dt = v \\ dv/dt = -2y-3v \end{cases}$

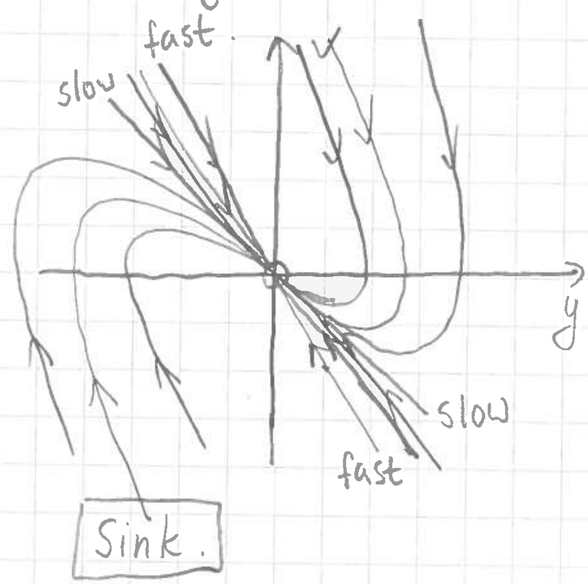
$p(\lambda)=\lambda^2+3\lambda+2=(\lambda+2)(\lambda+1)$

$\lambda_1=-2$ $\qquad\qquad$ $\lambda_2=-1$

$+2u+v=0$ $\qquad\qquad$ $u+v=0$

$V_1=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\qquad\qquad$ $V_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

fast $\qquad\qquad$ slow



#3) a) $p(\lambda)=\det\begin{pmatrix} 4-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix}=(4-\lambda)^2+1=0 \Rightarrow$

$4-\lambda=\pm i$, $\boxed{\lambda_1=4+i, \quad \lambda_2=4-i}$

$\lambda_1=4+i$ $\qquad\qquad$ $\lambda_2=4-i$

$[4-(4+i)]u+v=0$ $\qquad\qquad$ $[4-(4-i)]u+v=0$

$-iu+v=0$ $\qquad\qquad$ $iu+v=0$

$V_1=\begin{pmatrix} 1 \\ i \end{pmatrix}$ $\qquad\qquad$ $V_2=\begin{pmatrix} 1 \\ -i \end{pmatrix}$

b) The general solution in complex form is

$Y(t)=k_1e^{\lambda_1 t}V_1+k_2e^{\lambda_2 t}V_2=k_1e^{(4+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}+k_2e^{(4-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

c) $Y(0)=k_1\begin{pmatrix} 1 \\ i \end{pmatrix}+k_2\begin{pmatrix} 1 \\ -i \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} k_1+k_2=1 \\ ik_1-ik_2=2 \quad |\cdot i \end{cases}$

$\Rightarrow \begin{cases} k_1+k_2=1 \\ -k_1+k_2=2i \end{cases} \Rightarrow k_2=\dfrac{1+2i}{2}, \quad k_1=1-\dfrac{1+2i}{2}=\dfrac{1-2i}{2}.$

Answer: $Y(t)=\dfrac{1-2i}{2}e^{(4+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}+\dfrac{1+2i}{2}e^{(4-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$