Homework #13 (Solutions)

#1 (a) Eigenvalues: characteristic polynomial

is $p(\lambda) = \det\begin{pmatrix} 2-\lambda & 0 \\ 0 & -3-\lambda \end{pmatrix} = (2-\lambda)(-3-\lambda)$

$p(\lambda) = 0 \Rightarrow \lambda_1 = 2, \ \lambda_2 = -3$ are eigenvalues.

Eigenvectors: If $\lambda_1 = 2$, eigenvector solves the

system $\begin{cases} (2-2)u + 0v = 0 \\ 0 + (-3-2)v = 0 \end{cases} \Leftrightarrow \begin{cases} 0 = 0 \\ -5v = 0 \end{cases} \Leftrightarrow v = 0$

Eigenvector $V_1 = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

If $\lambda_2 = 3$, eigenvector solves

$\begin{cases} (2-3)u = 0 \\ 0 = 0 \end{cases} \Rightarrow u = 0 \Rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Answer: $\lambda_1 = 2, \ \lambda_2 = -3$

$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

b) $p(\lambda) = \det\begin{pmatrix} a-\lambda & 0 \\ 0 & c-\lambda \end{pmatrix} = (a-\lambda)(c-\lambda) = 0$

E-values are $\lambda_1 = a, \ \lambda_2 = c$.

If $\lambda_1 = a \Rightarrow \begin{cases} 0 = 0 \\ (c-a)v = 0 \end{cases}$ There are two cases:

- if $c \neq a \Rightarrow v = 0 \Rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- if $c = a \Rightarrow$ second equation becomes $0 = 0 \Rightarrow$ any non-zero vector is an eigenvector.

If $\lambda_2 = c, \ a \neq c \Rightarrow \begin{cases} (a-c)u = 0 \\ 0 = 0 \end{cases} \Rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Answer: $\lambda_1 = a, \ \lambda_2 = c$

- If $a \neq c$, then $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- If $a = c$, then any vector is an eigenvector.

(c) $p(\lambda) = \det\begin{pmatrix} 1-\lambda & 2 \\ 0 & 5-\lambda \end{pmatrix} = (1-\lambda)(5-\lambda) = 0$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 5$

Eigenvectors:

$\boxed{\begin{matrix}\lambda_1 = 1 \\ V_1 = \begin{pmatrix}1\\0\end{pmatrix}\end{matrix}} \Rightarrow \begin{cases} 0u + 2v = 0 \\ 0u + 4v = 0 \end{cases} \Rightarrow v = 0 \Rightarrow V_1 = \begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$\boxed{\begin{matrix}\lambda_2 = 5 \\ V_2 = \begin{pmatrix}1\\2\end{pmatrix}\end{matrix}} \Rightarrow \begin{cases} (1-5)u + 2v = 0 \\ 0u + (5-5)v = 0 \end{cases} \Rightarrow \begin{cases} -4u + 2v = 0 \\ 0 = 0 \end{cases} \Rightarrow v = 2u$

(d) $p(\lambda) = \det\begin{pmatrix} a-\lambda & b \\ 0 & c-\lambda \end{pmatrix} = (a-\lambda)(c-\lambda) = 0$.

Two cases: 1) $a \neq c$, then $\lambda_1 = a$, $\lambda_2 = c$

$\boxed{\begin{matrix}\lambda_1 = 0 \\ V_1 = \begin{pmatrix}1\\0\end{pmatrix}\end{matrix}} \Rightarrow \begin{cases} bv = 0 \\ (c-a)v = 0 \end{cases} \Rightarrow v = 0$

$\boxed{\begin{matrix}\lambda_2 = c \\ V_2 = \begin{pmatrix}b\\a-c\end{pmatrix}\end{matrix}} \Rightarrow \begin{cases} (a-c)u + bv = 0 \\ 0 = 0 \end{cases} \Rightarrow v = \frac{a-c}{b}u$

2) If $a = c$, then $\lambda_1 = \lambda_2 = a$ and there is only one linearly independent eigenvector $V_1 = \begin{pmatrix}1\\0\end{pmatrix}$.

e) $p(\lambda) = \begin{pmatrix} 0-\lambda & 1 \\ -5 & -6-\lambda \end{pmatrix} = -\lambda(-6-\lambda) + 5 = \lambda^2 + 6\lambda + 5 =$

$= (\lambda+5)(\lambda+1) = 0 \Rightarrow \lambda_1 = -5$, $\lambda_2 = -1$.

If $\boxed{\begin{matrix}\lambda_1 = -5 \\ V_1 = \begin{pmatrix}1\\-5\end{pmatrix}\end{matrix}} \Rightarrow \begin{cases} 5u + v = 0 \\ -5u - v = 0 \end{cases} \Rightarrow v = -5u$

$$\lambda_2 = -1 \Rightarrow \begin{cases} u + v = 0 \\ -5u - 5v = 0 \end{cases} \Rightarrow v = -u$$

$$V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(Remark: Notice that we have two straight line (basic) solutions $Y_1(t) = \begin{pmatrix} y_1(t) \\ v_1(t) \end{pmatrix} = e^{\lambda_1 t} V_1 = e^{-5t}\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ and $Y_2(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $y_1' = v$, $y_2' = v_2$, as expected.)

#2. $AV = \lambda V$,

$$\begin{pmatrix} a & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \begin{cases} a - 2 = \lambda \\ 2 - 6 = -2\lambda \end{cases}$$

$$\Rightarrow \boxed{\lambda = 2, \quad a = 4}$$

#3. a) $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$, $p(\lambda) = \det\begin{pmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix}$

$$= (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$$

$$\boxed{\begin{matrix} \lambda_1 = 2 \\ V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{matrix}} \Rightarrow \begin{cases} (4-2)u - 2v = 0 \\ u + (1-2)v = 0 \end{cases} \Rightarrow u = v$$

$$\boxed{\begin{matrix} \lambda_2 = 3 \\ V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{matrix}} \Rightarrow \begin{cases} (4-3)u - 2v = 0 \\ u + (1-3)v = 0 \end{cases} \Rightarrow u = 2v$$

General solution: $Y(t) = k_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b) $p(\lambda) = \begin{pmatrix} 2-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - 4$

$$= \lambda^2 - 3\lambda - 2 = 0$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9+8}}{2}$$

$$\lambda_1 = \frac{3+\sqrt{17}}{2} \Rightarrow \begin{cases} \left(2-\frac{3+\sqrt{17}}{2}\right)u + 2v = 0 \\ 2u + \left(1-\frac{3+\sqrt{17}}{2}\right)v = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1-\sqrt{17}}{2}u + 2v = 0 \\ 2u + \frac{-1-\sqrt{17}}{2}v = 0 \end{cases} \Rightarrow v = -\frac{1-\sqrt{17}}{4}u$$

$$V_1 = \begin{pmatrix} 4 \\ -1+\sqrt{17} \end{pmatrix} \quad \left(\text{Other possible answer } V_1 = \begin{pmatrix} 1+\sqrt{17} \\ 4 \end{pmatrix}\right)$$

$$\lambda_2 = \frac{3-\sqrt{17}}{2} \Rightarrow \begin{cases} \left(2-\frac{3-\sqrt{17}}{2}\right)u + 2v = 0 \\ 2u + \left(1-\frac{3-\sqrt{17}}{2}\right)v = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \left(\frac{1+\sqrt{17}}{2}\right)u + 2v = 0 \\ 2u + \left(\frac{-1+\sqrt{17}}{2}v\right) = 0 \end{cases}$$

$$V_2 = \begin{pmatrix} 4 \\ -1-\sqrt{17} \end{pmatrix} \quad \left(\text{or } V_2 = \begin{pmatrix} 1-\sqrt{17} \\ 4 \end{pmatrix}\right)$$