Total is 37pts: Handout to homework # 11 (Solutions) ①

1 a) Plugging in: $(e^{-t})'' + 5(e^{-t})' + 4e^{-t}$

$= e^{-t} - 5e^{-t} + 4e^{-t} = 0$

Similarly, $(e^{-4t})'' + 5(e^{-4t})' + 4e^{-4t} =$

2/2 $= 16e^{-4t} - 20e^{-4t} + 4e^{-4t} = 0.$

b) Plugging in:

$(C_1 e^{-t} + C_2 e^{-4t})'' + 5(C_1 e^{-t} + C_2 e^{-4t})'$

$+ 4(C_1 e^{-t} + C_2 e^{-4t}) =$

$= C_1[(e^{-t})'' + 5(e^{-t})' + 4e^{-t}] +$

$+ C_2[(e^{-4t})'' + 5(e^{-4t})' + 4e^{-4t}] \overset{\text{from a)}}{=}$

$= C_1 \cdot 0 + C_2 \cdot 0 = 0.$

5/5

c) This is the system associated to $y'' + 5y' + 4y$. So any solution $y(t)$ of this differential equation, leads to a solution $\vec{Y}(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$ of the system.

The solution from b), leads to

5/5 $Y(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{-t} + C_2 e^{-4t} \\ (C_1 e^{-t} + C_2 e^{-4t})' \end{pmatrix} =$

$= \begin{pmatrix} C_1 e^{-t} + C_2 e^{-4t} \\ -C_1 e^{-t} - 4C_2 e^{-4t} \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$

(2)

d) $Y(0) = C_1\begin{pmatrix}1\\-1\end{pmatrix} + C_2\begin{pmatrix}1\\-4\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$

We want $\begin{cases} C_1 + C_2 = 3 \\ -C_1 - 4C_2 = -2 \end{cases}$

We add two equations: $-3C_2 = 1$, $C_2 = -\frac{1}{3}$

$\Rightarrow C_1 = 3 - C_2 = 3 + \frac{1}{3} = \frac{10}{3}$.

Answer: $Y(t) = \frac{10}{3}e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} - \frac{1}{3}e^{-4t}\begin{pmatrix}1\\-4\end{pmatrix}$,

or $y(t) = \frac{10}{3}e^{-t} - \frac{1}{3}e^{-4t}$

$v(t) = -\frac{10}{3}e^{-t} + \frac{4}{3}e^{-4t}$.

3/3

2a) We plug in:

$\frac{d}{dt}\left(e^{\lambda_1 t}\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}2 & 2\\1 & 3\end{pmatrix}\left(e^{\lambda_1 t}\begin{pmatrix}1\\1\end{pmatrix}\right)$

$\lambda_1 \cancel{e^{\lambda_1 t}}\begin{pmatrix}1\\1\end{pmatrix} = \cancel{e^{\lambda_1 t}}\begin{pmatrix}2 & 2\\1 & 3\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$

$\lambda_1\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix} = 4\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \boxed{\lambda_1 = 4}$

Similarly, $\lambda_2\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}2 & 2\\1 & 3\end{pmatrix}\begin{pmatrix}-2\\1\end{pmatrix}$,

5/5 $\lambda_2\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix} \Rightarrow \boxed{\lambda_2 = 1.}$

b) The system has two solutions that are not multiples of each other

$Y_1(t) = e^{4t}\begin{pmatrix}1\\1\end{pmatrix}$ and $Y_2(t) = e^{t}\begin{pmatrix}-2\\1\end{pmatrix}$.

By the Linearity Principle,

the general solution is (3)

$$Y(t) = c_1 Y_1(t) + c_2 Y_2(t) = c_1 e^{4t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{t}\begin{pmatrix}-2\\1\end{pmatrix}$$

3/3

c) $Y(0) = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}$,

thus we want

$$\begin{cases} c_1 - 2c_2 = 4 \\ c_1 + c_2 = 3 \end{cases}$$

Subtracting second equation from first, we obtain

$$-3c_2 = 1, \quad \boxed{c_2 = -1/3}$$

$$\boxed{c_1 = 3 - c_2 = 10/3}$$

3/3

Thus, $Y(t) = 10/3\, e^{4t}\begin{pmatrix}1\\1\end{pmatrix} - 1/3\, e^{t}\begin{pmatrix}-2\\1\end{pmatrix}$

d) $Y(0) = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$,

$$\begin{cases} c_1 - 2c_2 = a \\ c_1 + c_2 = b. \end{cases}$$

Subtracting as before, we get

$$-3c_2 = a - b, \quad \boxed{c_2 = \frac{1}{3}(b-a)}$$

3/3 $c_1 = b - c_2 = b - 1/3(b-a) = 2/3\, b + 1/3\, a$

Thus, $Y(t) = (2/3\, b + 1/3\, a)e^{4t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{3}(b-a)e^{t}\begin{pmatrix}-2\\1\end{pmatrix}$

3) a)

$$A(\vec{X}+\vec{Y}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} =$$

$$= \begin{pmatrix} a(x_1+y_1) + b(x_2+y_2) \\ c(x_1+y_1) + d(x_2+y_2) \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$$

$$+ \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

5|5 $= A\vec{X} + B\vec{Y}.$

b) $$A(kX) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} kx_1 \\ kx_2 \end{pmatrix}$$

$$= \begin{pmatrix} akx_1 + bkx_2 \\ ckx_1 + dkx_2 \end{pmatrix} = k\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$$

3|3

$$= k\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$