1

Solutions to homework #15 (Sec 3.6)

10. The characteristic polynomial is

$$s^2 + 4s + 20,$$

so the eigenvalues are $s = -2 \pm 4i$. Hence, the general solution is

$$y(t) = k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t.$$

From the initial condition $y(0) = 2$, we see that $k_1 = 2$. Differentiating

$$y(t) = 2e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = -4 + 4k_2$. Since $y'(0) = -8$, we have $k_2 = -1$. Hence, the solution to our initial-value problem is

$$y(t) = 2e^{-2t} \cos 4t - e^{-2t} \sin 4t.$$

13. **(a)** The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 7y = 0,$$

and the corresponding system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -7y - 8v.$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$\lambda^2 + 8\lambda + 7 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -7$.

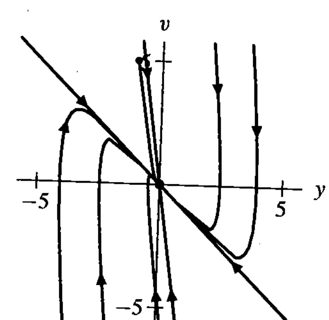
To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = -y \\ -7y - 8v = -v. \end{cases}$$

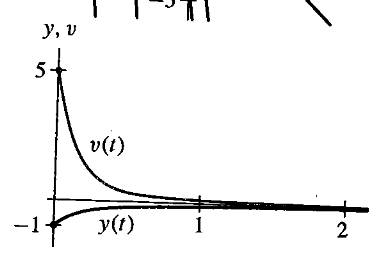
From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = -y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -7$ must satisfy the equation $v = -7y$.

(c) Since the eigenvalues are real and negative, the equilibrium point at the origin is a sink, and the system is overdamped.

(d) We know that all solution curves approach the origin as $t \to \infty$ and, with the exception of those whose initial conditions lie on the line $v = -7y$, these solution curves approach the origin tangent to the line $v = -y$.



(e) From the phase portrait, we see that $y(t)$ increases monotonically toward 0 as $t \to \infty$. Also, $v(t)$ decreases monotonically toward 0. It is useful to remember that $v = dy/dt$.



14. **(a)** The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 0,$$

and the corresponding system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -8y - 6v.$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$\lambda^2 + 6\lambda + 8 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -2$.

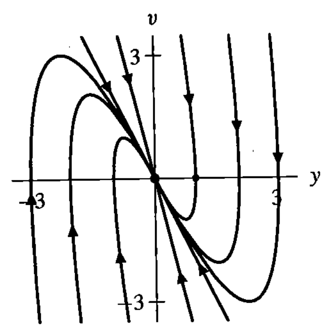
To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = -4y \\ -8y - 6v = -4v. \end{cases}$$

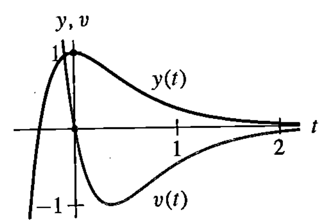
From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = -4y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -2$ must satisfy the equation $v = -2y$.

(c) Since the eigenvalues are real and negative, the equilibrium point at the origin is a sink, and the system is overdamped.

(d) We know that all solution curves approach the origin as $t \to \infty$ and, with the exception of those whose initial conditions lie on the line $v = -4y$, these solution curves approach the origin tangent to the line $v = -2y$.



(e) From the phase portrait, we see that $v(t)$ initially decreases from 0 and then increases and tends toward 0 as $t \to \infty$. Also, $y(t)$ decreases monotonically toward 0. It is useful to remember that $v = dy/dt$.



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16. **(a)** The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 8y = 0,$$

and the corresponding system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -8y.$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$\lambda^2 + 8 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 2\sqrt{2}\,i$ and $\lambda_2 = -2\sqrt{2}\,i$.

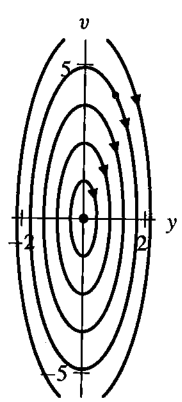
To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = 2\sqrt{2}\,iy \\ -8y = 2\sqrt{2}\,iv. \end{cases}$$

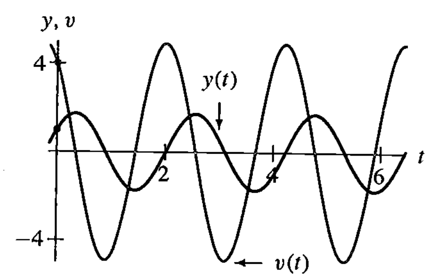
From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = 2\sqrt{2}\,iy$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -2\sqrt{2}\,i$ must satisfy the equation $v = -2\sqrt{2}\,iy$.

(c) Since the eigenvalues are pure imaginary the equilibrium point at the origin is a center with natural period $\pi/\sqrt{2}$, and the system is undamped.

(d) All solutions move clockwise.



(e) Each graph is periodic with period $\pi/\sqrt{2}$.



(4)

22. **(a)** The second-order equation is

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 0,$$

so the characteristic equation is

$$s^2 + 6s + 8 = 0.$$

The roots are $s = -4$ and $s = -2$. The general solution is

$$y(t) = k_1 e^{-4t} + k_2 e^{-2t}.$$

(b) To find the particular solution we compute

$$v(t) = -4k_1 e^{-4t} - 2k_2 e^{-2t}.$$

The particular solution satisfies

$$\begin{cases} 1 = y(0) = k_1 + k_2 \\ 0 = v(0) = -4k_1 - 2k_2. \end{cases}$$

The first equation yields $k_1 = -k_2 + 1$. Substituting into the second we obtain $0 = 2k_2 - 4$, which implies $k_2 = 2$. The first equation then yields $k_1 = -1$. The particular solution is

$$y(t) = -e^{-4t} + 2e^{-2t}.$$

(c) The $y(t)$- and $v(t)$-graphs are displayed in the solution of Exercise 14.

30. Note that

$$\frac{dy}{dt} = \frac{d}{dt}(k_1 y_1 + k_2 y_2) = k_1 \frac{dy_1}{dt} + k_2 \frac{dy_2}{dt}$$

and

$$\frac{d^2y}{dt^2} = \frac{d^2}{dt^2}(k_1 y_1 + k_2 y_2) = k_1 \frac{d^2y_1}{dt^2} + k_2 \frac{d^2y_2}{dt^2}.$$

Therefore,

$$\begin{aligned} \frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy &= k_1 \frac{d^2y_1}{dt^2} + k_2 \frac{d^2y_2}{dt^2} + p\left(k_1 \frac{dy_1}{dt} + k_2 \frac{dy_2}{dt}\right) + q\,(k_1 y_1 + k_2 y_2) \\ &= k_1 \left(\frac{d^2y_1}{dt^2} + p\frac{dy_1}{dt} + qy_1\right) + k_2 \left(\frac{d^2y_2}{dt^2} + p\frac{dy_2}{dt} + qy_2\right) \\ &= 0. \end{aligned}$$

31. Note that

$$\frac{dy}{dt} = \frac{d}{dt}(y_{\text{re}} + i y_{\text{im}}) = \frac{dy_{\text{re}}}{dt} + i\frac{dy_{\text{im}}}{dt}$$

and

$$\frac{d^2y}{dt^2} = \frac{d^2}{dt^2}(y_{\text{re}} + i y_{\text{im}}) = \frac{d^2y_{\text{re}}}{dt^2} + i\frac{d^2y_{\text{im}}}{dt^2}.$$

Then note that

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \left(\frac{d^2y_{\text{re}}}{dt^2} + p\frac{dy_{\text{re}}}{dt} + qy_{\text{re}}\right) + i\left(\frac{d^2y_{\text{im}}}{dt^2} + p\frac{dy_{\text{im}}}{dt} + qy_{\text{im}}\right).$$

Both

$$\frac{d^2y_{\text{re}}}{dt^2} + p\frac{dy_{\text{re}}}{dt} + qy_{\text{re}} = 0 \quad \text{and} \quad \frac{d^2y_{\text{im}}}{dt^2} + p\frac{dy_{\text{im}}}{dt} + qy_{\text{im}} = 0$$

because a complex number is zero only if both its real and imaginary parts vanish. In other words, $y_{\text{re}}(t)$ and $y_{\text{im}}(t)$ are solutions of the original equation.