Handout for Sec. 3.6

REVIEW SHEET FOR TEST 3

The best way to prepare for a test is to review homework problems, examples from the textbook and class notes. Make sure that you can solve these problems in a reasonable amount of time without reference to the textbook or class notes. It is important that on the test you show all your work and explain your answers. Just answers (especially wrong ones!) without any explanation will earn you no credit.

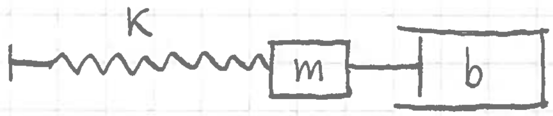
List of major topics covered in class.

1. Section 3.1. Linear systems: matrix notation (pp. 243 -246). Linearity principle (pp. 249- 255). Applications of Linearity principle to solving initial value problems (pp. 252- 255). Linear independence and general solution (pp. 255-256). Undamped harmonic oscillator (pp. 256 - 258). #5, 9, 11, 17, 24, 25, 27, 30 after section 3.1.
2. Section 3.2. Straight-line solutions, equation for finding straight-line solutions, definition of eigenvalues and associate eigenvectors, lines of eigenvectors. Computation of eigenvalues and eigenvectors, characteristic polynomial, formulas for straight-line solutions. Linear independence of eigenvectors corresponding to distinct eigenvalues. #1, 2, 3, 7, 8, 11, 13, 16, 21, 23 after section 3.2.
3. Section 3.3. Geometry of a phase plane for linear systems with real eigenvalues: saddles, sinks, and sources. Stability of equilibrium point (0,0). You should be able to draw an approximate phase portrait of a system from the information about its eigenvalues and eigenvectors (pp. 270-291, class notes, handouts). You also should be able to draw approximate graphs of x(t) and y(t) from a given solution curve. # 1, 2, 3, 5, 6, 9, 11, 13, 14 after section 3.3.
4. Section 3.4. Systems with complex eigenvalues: Euler formula, obtaining two real solutions from a complex solution. General solution in complex form and in real form. Theorems from class: complex eigenvalues and eigenvectors of a real matrix come in conjugate pairs, and real and imaginary parts of a complex solution are also solutions. Spiral sources, spiral sinks, centers. Natural frequency, natural period. Direction of spiraling: clockwise or counterclockwise. Stability of equilibrium point. You should be able to determine the type of equilibrium point and draw an approximate phase portrait by computing eigenvalues of the system (section 3.4, class notes, handouts). # 1, 2, 3, 4, 5, 6, 8, 10, 11, 14, 17, 22 (see handout) after section 3.4.
5. Section 3.6. Second-order linear equations - two methods of solutions: guess-and- test and reduction to a system. Classification of harmonic oscillators: undamped, underdamped, critically damped, and overdamped oscillators. For each type of oscillators you need to know: eigenvalue condition, condition on the discriminant D=b^2-4mk corresponding to this type, typical phase-plane portraits and y(t), v(t) graphs (section 3.6 and class notes). # 9, 11, 16, 23, 24, 30, 31 after section 3.6.

GOOD LUCK!

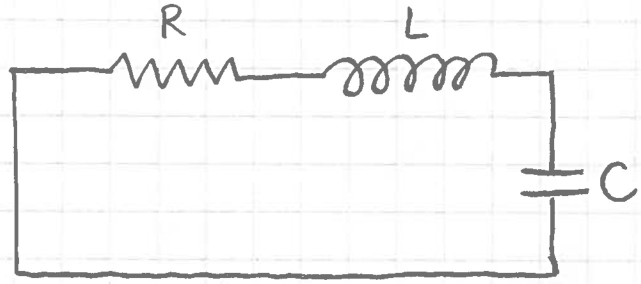
Comparison between harmonic oscillators and R-L-C circuits.

Harmonic oscillator (mass - spring - dashpot)



$$m \frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t)$$

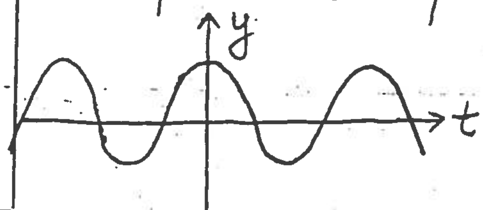
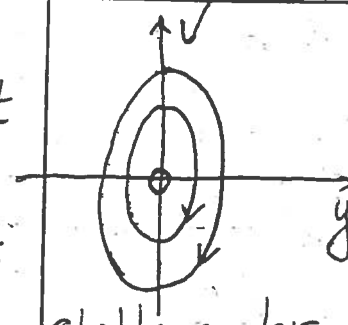
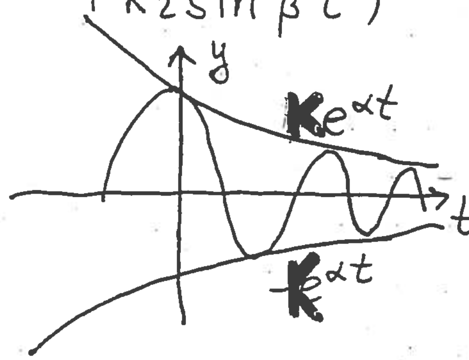
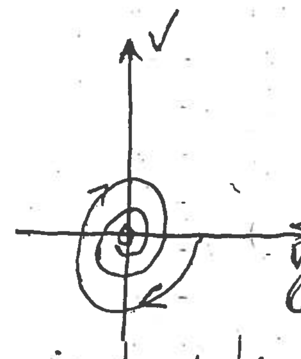
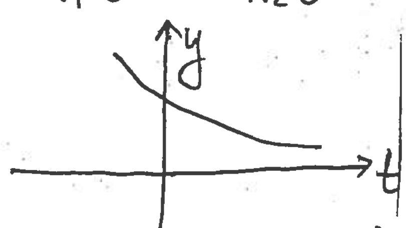
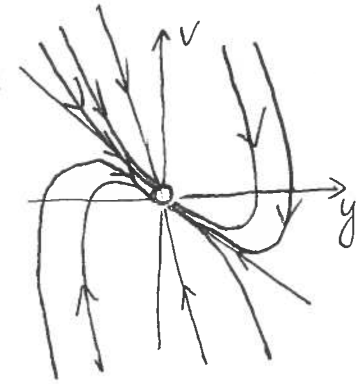
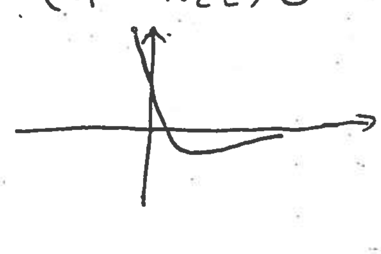
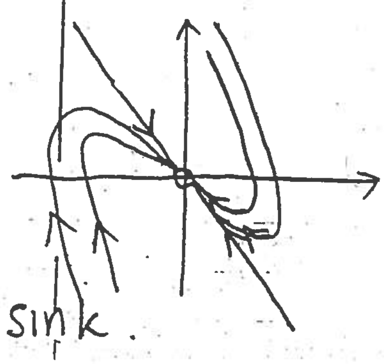
Resistor - inductor - capacitor circuit



$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

Harmonic oscillator	R-L-C circuit
$y(t)$ = displacement from equilibrium in meters at time t	$q(t)$ = charge at time t on the capacitor (coulombs)
m = mass (kilograms)	L = inductance (henrys)
b = friction coefficient (newtons · sec/meter)	R = resistance (ohms)
k = spring coefficient $\left(\frac{\text{newtons}}{\text{meter}}\right)$	$\frac{1}{C} = \frac{1}{\text{capacitance}}$ $\left(\frac{1}{\text{farads}}\right)$
$v(t)$ = velocity (m/sec) $v(t) = \frac{dy}{dt}$	$I(t)$ = current at time t $I(t) = \frac{dq}{dt}$ (amperes)
$f(t)$ = external force (Newtons)	$V(t)$ = voltage (Volts)

Classification of harmonic oscillators (or RLC circuits).

Name of the case	Eigenvalues (roots of $p(\lambda)$)	General solution and typical solution graph	Phase portrait
No damping $b=0$ (no friction)	$\lambda_1 = i\beta$ $\lambda_2 = -i\beta$	$k_1 \cos\beta t + k_2 \sin\beta t$ 	 stable center
Underdamping $b^2 - 4mk < 0$ (small friction)	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ $\alpha < 0,\ \beta \neq 0$	$e^{\alpha t}(k_1 \cos\beta t +$ $+ k_2 \sin\beta t)$ 	 spiral sink
Overdamping $b^2 - 4mk > 0$ (Large friction)	$\lambda_1 < 0,\ \lambda_2 < 0$	$k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$ 	 sink.
Critical damping $b^2 - 4mk = 0$ (critical friction)	$\lambda_1 = \lambda_2 = \lambda$ $= -\frac{b}{2m}$	$(k_1 + k_2 t)e^{\lambda t}$ 	 sink.

Two methods of solving $my'' + by' + ky = 0$.

Handout for section 3.6

Differential equation	System
$my'' + by' + ky = 0$	$\begin{cases} dy/dt = v, \\ \frac{dv}{dt} = -\frac{k}{m}y - \frac{b}{m}v \end{cases}$ $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}$
Seek solution in the form $y = e^{\lambda t}$, then λ satisfies $m\lambda^2 + b\lambda + k = 0$ (*)	Eigenvalues satisfy $p(\lambda) = \det\begin{pmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{b}{m} - \lambda \end{pmatrix} = \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$ (same equation as (*)) Eigenvectors: $V_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

If λ_1 and λ_2 are real with $\lambda_1 \neq \lambda_2$, general solution is

$y(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$	$Y(t) = k_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + k_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

If $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, general solution in real form is

$y(t) = e^{\alpha t}(k_1 \cos(\beta t) + k_2 \sin(\beta t))$	$Y(t) = k_1 e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ \alpha\cos(\beta t) - \beta\sin(\beta t) \end{pmatrix} + k_2 e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \alpha\sin(\beta t) + \beta\cos(\beta t) \end{pmatrix} = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix}$

Relation: $y(t) = y(t)$, $y'(t) = v(t)$.

Handout on linear combination of trig functions

Linear combination of trigonometric functions

$$K_1 \cos(\beta t) + K_2 \sin(\beta t)$$ can be written as

a single function

$$K \cos(\beta t - \phi),$$

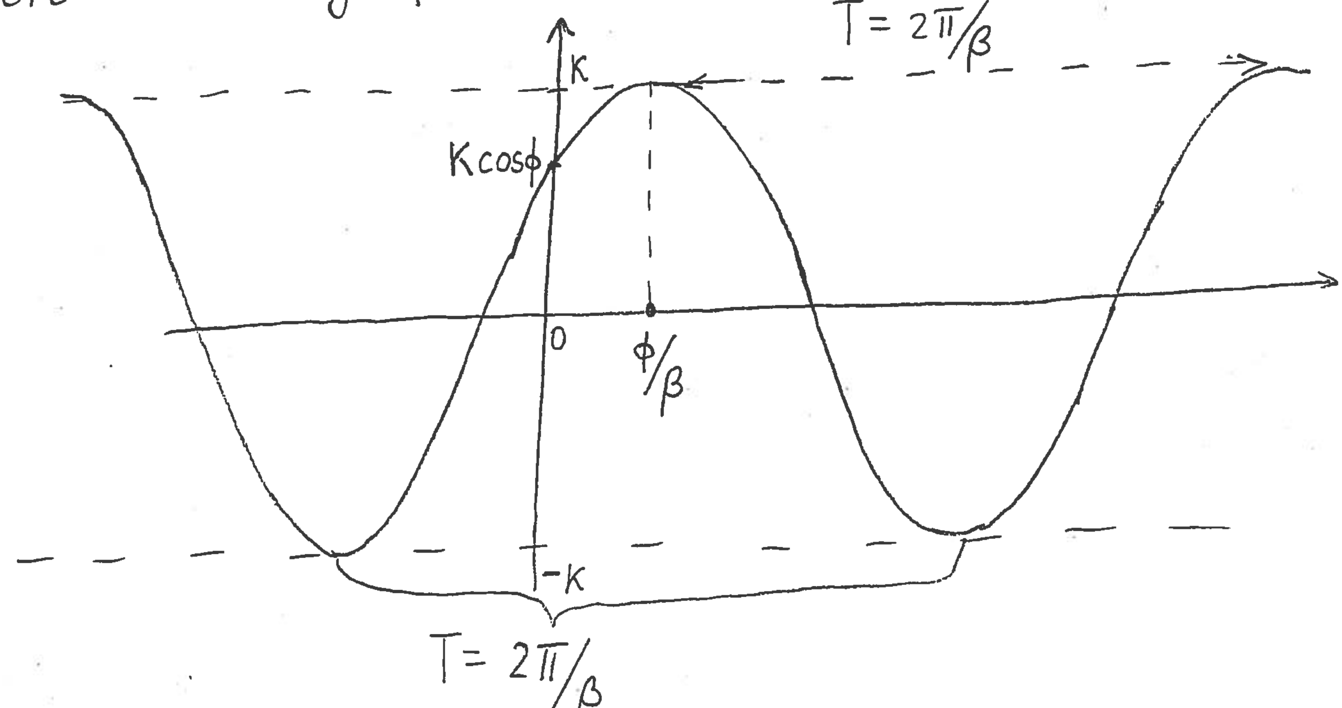
where $K = \sqrt{K_1^2 + K_2^2}$ is called <u>amplitude</u>

and ϕ = <u>phase</u> of mechanical motion or electrical signal represented by this function.

Phase ϕ can be found from relations

$$\cos\phi = \frac{K_1}{K} \quad \text{and} \quad \sin\phi = \frac{K_2}{K}.$$

Here is the graph of $K\cos(\beta t - \phi) = K\cos\left(\beta\left(t - \frac{\phi}{\beta}\right)\right)$



Proof of the formula:

We use trigonometric formula

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

to expand as

$$K\cos(\beta t - \phi) = K\cos(\beta t)\cos\phi + K\sin(\beta t)\sin\phi.$$

Comparing this expansion to $K_1\cos(\beta t) + K_2\sin(\beta t)$ we conclude that

$$K_1 = K\cos\phi \quad \text{and} \quad K_2 = K\sin\phi.$$

We can solve for K as follows:

$$K_1^2 + K_2^2 = K^2(\cos^2\phi + \sin^2\phi) = K^2, \text{ so}$$

$$K = \sqrt{K_1^2 + K_2^2}.$$

If K is known, angle ϕ can be found from relations

$$\cos\phi = \frac{K_1}{K} \quad \text{and} \quad \sin\phi = \frac{K_2}{K}.$$