Real form of general solution for systems with complex eigenvalues.

Step 1 Find complex eigenvalue λ and corresponding complex eigenvector V.
Form complex solution $Y(t) = e^{\lambda t} V$.

Step 2 Use Euler formula $e^{a+bi} = e^{a}(\cos b + i \sin b)$ to write $Y(t) = Y_{re}(t) + i\, Y_{im}(t)$, where real vector-valued functions $Y_{re}(t)$ and $Y_{im}(t)$ are real and imaginary parts of the complex solution $Y(t)$.

Step 3. Form the general solution

$$k_1 Y_{re}(t) + k_2 Y_{im}(t)$$

(this is the real form of the general solution)

Fact: If $Y(t) = Y_{re}(t) + i\, Y_{im}(t)$ is a solution to a system $\frac{dY}{dt} = AY$ with real matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $Y_{re}(t)$ and $Y_{im}(t)$ are also solutions of the same system.

Proof. We are given that $\frac{dY}{dt} = AY$. Thus

$$\frac{d(Y_{re} + i\, Y_{im})}{dt} = A(Y_{re} + i\, Y_{im})$$

$= Y_{re}(t) + i\,Y_{im}(t)$, where

$$Y_{re}(t) = e^{-t}\begin{pmatrix} 2\cos 2t \\ -\sin 2t \end{pmatrix} \text{ and } Y_{im}(t) = e^{-t}\begin{pmatrix} -2\sin(2t) \\ -\cos(2t) \end{pmatrix}$$

are real and imaginary parts of complex solution.

<u>Theorem:</u> If $Y(t) = Y_{re}(t) + i\,Y_{im}(t)$ is a solution of a linear system $\frac{dY}{dt} = AY$ with real matrix A, then $Y_{re}(t)$ and $Y_{im}(t)$ are also solutions of the same system.

<u>Proof:</u> $\frac{dY_{re}}{dt} + i\frac{dY_{im}}{dt} = A\,Y_{re}(t) + i\,A\,Y_{im}(t)$,

compare real and imaginary parts on both sides.

▲

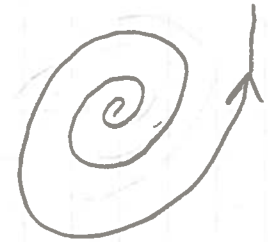
So general solution to $\frac{dY}{dt} = \begin{pmatrix} -1 & 4 \\ -1 & -1 \end{pmatrix} Y$ in real form is

$$Y(t) = k_1 e^{-t}\begin{pmatrix} 2\cos 2t \\ -\sin 2t \end{pmatrix} + k_2 e^{-t}\begin{pmatrix} -2\sin(2t) \\ -\cos(2t) \end{pmatrix}$$

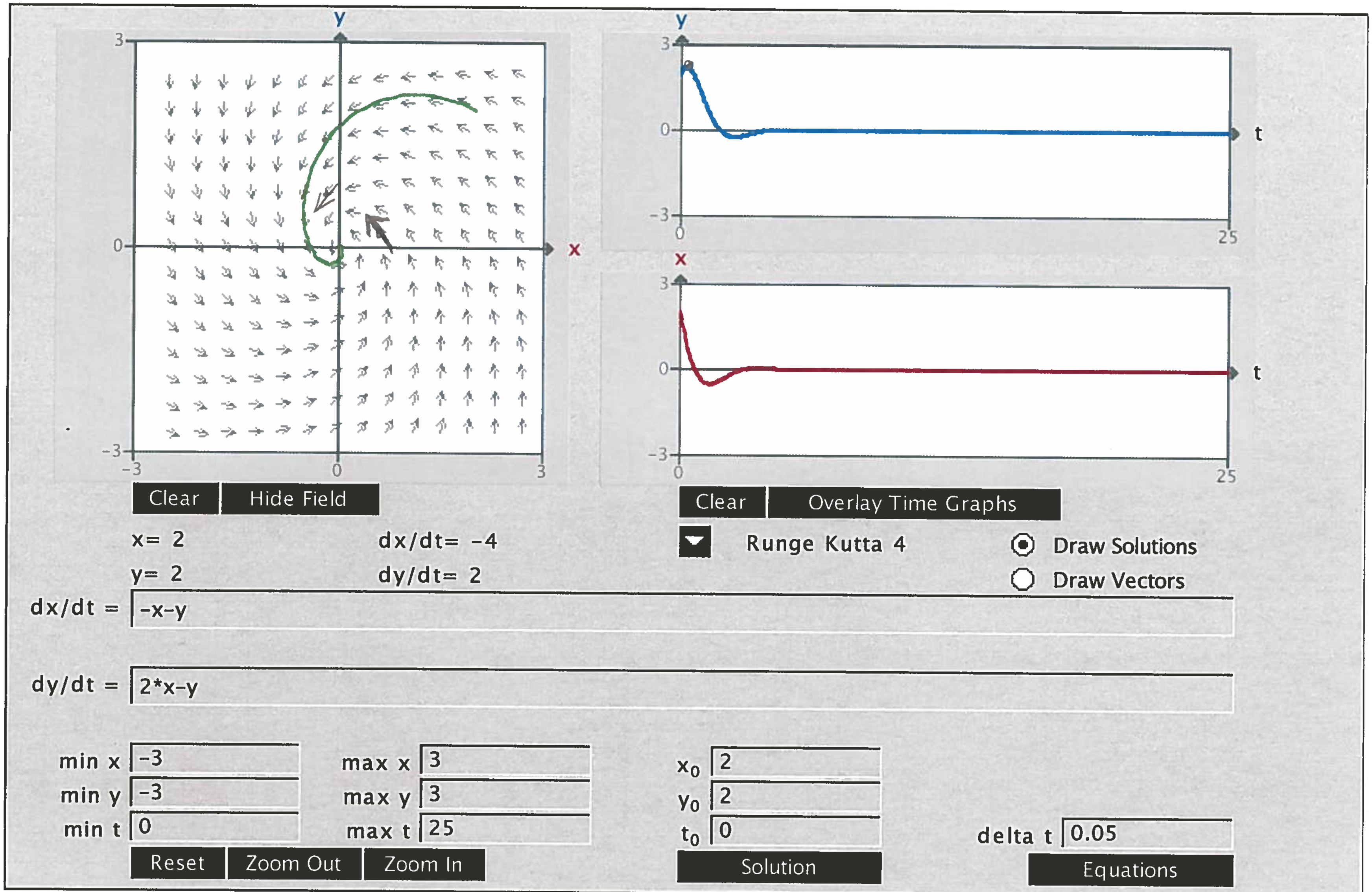
If $Y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = k_1\begin{pmatrix} 2 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ -1 \end{pmatrix}$,

so $k_1 = \frac{1}{2}$ and $k_2 = -1$.

(3) Classification of equlibrium point (0, 0) for the system $\frac{dY}{dt} = AY$, where matrix A has complex eigenvalues.

Eigenvalues	Name of eq. pt.	Stability	Pictures.
$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ α is positive	spiral source (or unstable focus)	unstable	 or 
$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ α is negative	spiral sink (stable focus)	stable	 or 
$\lambda_1 = i\beta$ $\lambda_2 = -i\beta$ (purely imaginary)	Center	stable	  Natural period : $\frac{2\pi}{\beta}$ Natural frequency: $\beta/2\pi$

Example 1.



$$\begin{cases} \dfrac{dx}{dt} = -x-y \\ \dfrac{dy}{dt} = 2x-y \end{cases}$$

$\lambda_1 = -1+\sqrt{2}i$

$\alpha = -1 \Rightarrow$ spiral sink

$\beta = \sqrt{2} \Rightarrow$ natural period $= \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

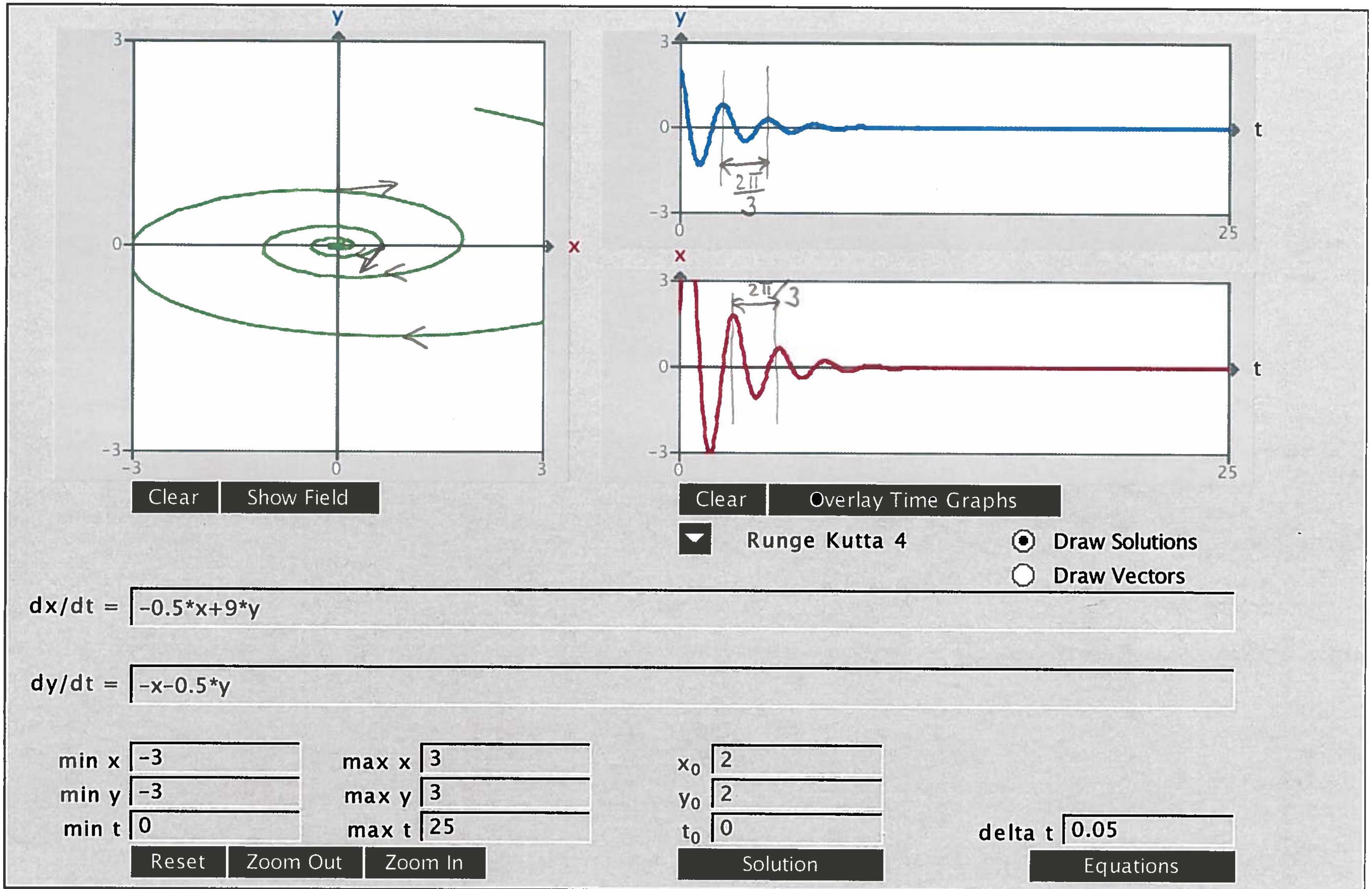
Counterclockwise rotation

$F(x,y) = (-x-y,\ 2x-y)$

$F(1,0) = (-1,2)$, $F(0,1) = (-1,-1)$

⑤ Small α, large β

Example 2.



$$\begin{cases} \frac{dx}{dt} = -0.5x + 9y \\ \frac{dy}{dt} = -x - 0.5y \end{cases}$$

$\alpha = -0.5 \Rightarrow$ spiral sink

$\beta = 3 \Rightarrow$ natural period $= \frac{2\pi}{3}$

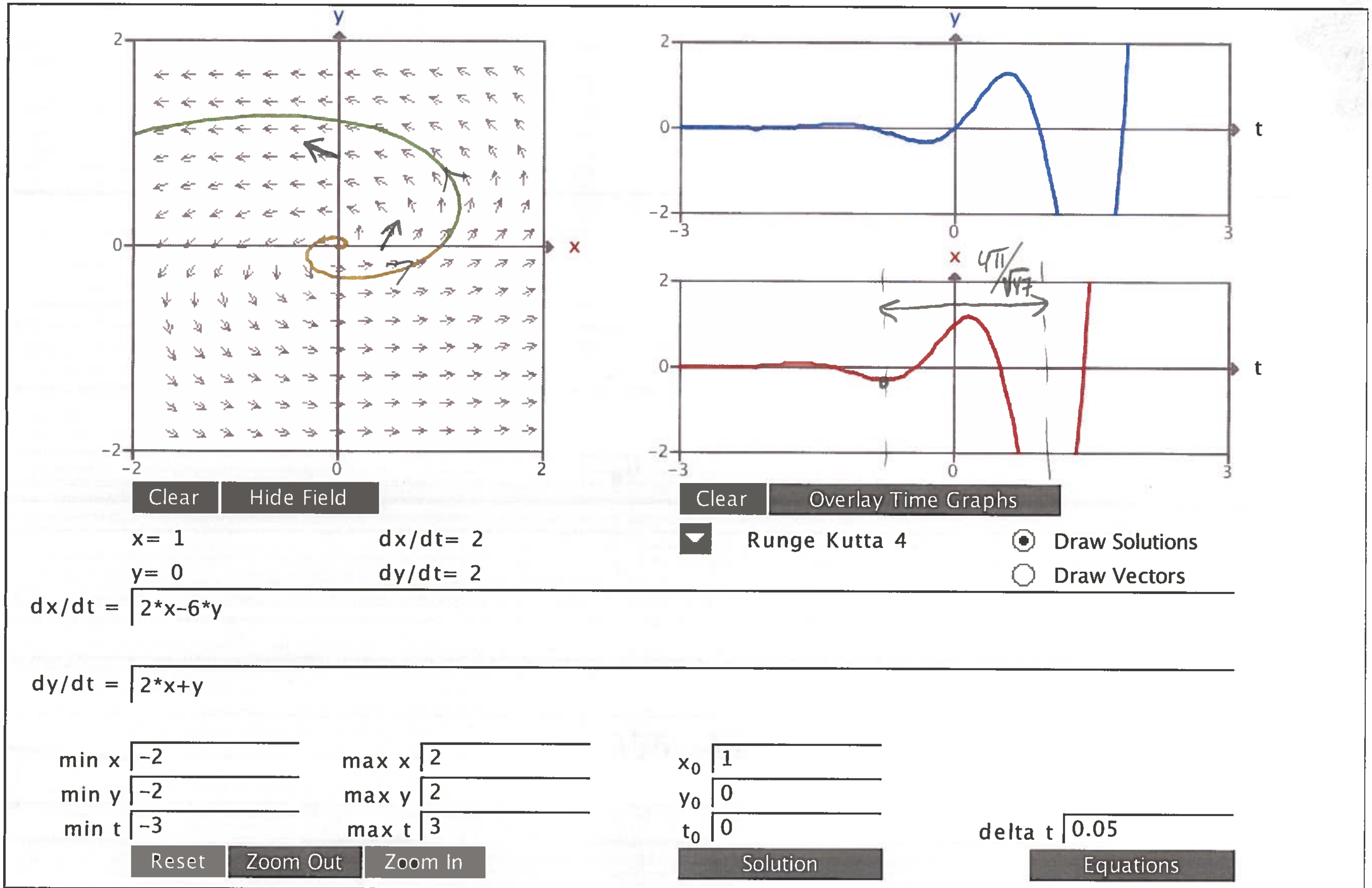
$\lambda_1 = -0.5 + 3i$

$F(x,y) = (0.5x + 9y, -x + 0.5y)$

$F(1,0) = (0.5, -1)$ $F(0,1) = (9, 0.5)$

Clockwise rotation

Example 3.



$$\frac{dx}{dt} = 2x - 6y$$

$$\frac{dy}{dt} = 2x + y$$

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{47}}{2} i$$

$\alpha = 3/2 \Rightarrow$ spiral source

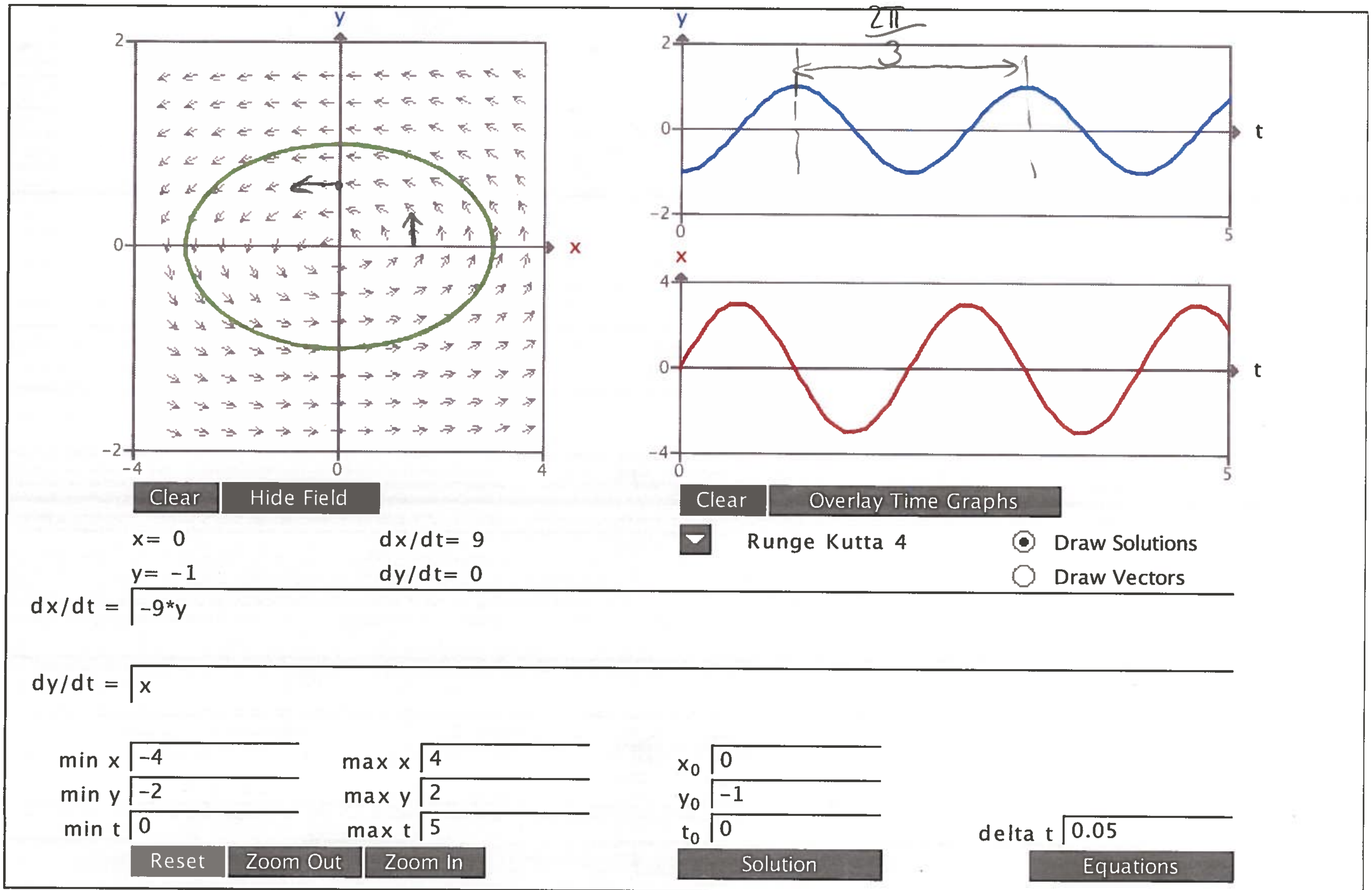
$\beta = \frac{\sqrt{47}}{2} \Rightarrow$ natural period $= \frac{4\pi}{\sqrt{47}}$

$F(x,y) = (2x-6y,\ 2x+y)$

$F(1,0) = (2,2)$ $\quad F(0,1) = (-6,1)$

Rotation counterclockwise

Example 4.



$$\begin{cases} \frac{dx}{dt} = -9y \\ \frac{dy}{dt} = x \end{cases}$$

$\lambda_1 = 3i$

$\alpha = 0 \Rightarrow$ center

$\beta = 3 \Rightarrow$ natural period $= \frac{2\pi}{3}$

rotation counterclockwise

$F(x,y) = (-9y, x)$, $F(1,0) = (0,1)$

$F(0,1) = (-9,0)$