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Real form of general solution for systems with complex eigenvalues.

Handout
for Section 3.4.

Step 1. Find complex eigenvalue λ and corresponding complex eigenvector V .

Form complex solution $Y(t) = e^{\lambda t} V$.

Step 2. Use Euler formula $e^{a+bi} = e^a(\cos bt + i \sin bt)$

to write $Y(t) = Y_{re}(t) + i Y_{im}(t)$, where real vector-valued functions $Y_{re}(t)$ and $Y_{im}(t)$ are real and imaginary parts of the complex solution $Y(t)$.

Step 3. Form the general solution

$$k_1 Y_{re}(t) + k_2 Y_{im}(t)$$

(this is the real form of the general solution)

Fact: If $Y(t) = Y_{re}(t) + i Y_{im}(t)$ is a solution to a system $\frac{dY}{dt} = AY$ with real matrix

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $Y_{re}(t)$ and $Y_{im}(t)$ are also solutions of the same system.

Proof: We are given that $\frac{dY}{dt} = AY$. Thus

$$\frac{d(Y_{re} + i Y_{im})}{dt} = A(Y_{re} + i Y_{im})$$

= $Y_{re}(t) + i Y_{im}(t)$, where

● $Y_{re}(t) = e^{-t} \begin{pmatrix} 2 \cos 2t \\ -\sin 2t \end{pmatrix}$ and $Y_{im}(t) = e^{-t} \begin{pmatrix} -2 \sin(2t) \\ -\cos(2t) \end{pmatrix}$

are real and imaginary parts of complex solution.

Theorem: If $Y(t) = Y_{re}(t) + i Y_{im}(t)$ is a solution of a linear system $\frac{dY}{dt} = AY$ with real matrix A , then $Y_{re}(t)$ and $Y_{im}(t)$ are also solutions of the same system.

Proof: $\frac{dY_{re}}{dt} + i \frac{dY_{im}}{dt} = A Y_{re}(t) + i A Y_{im}(t)$,
● compare real and imaginary parts on both sides.

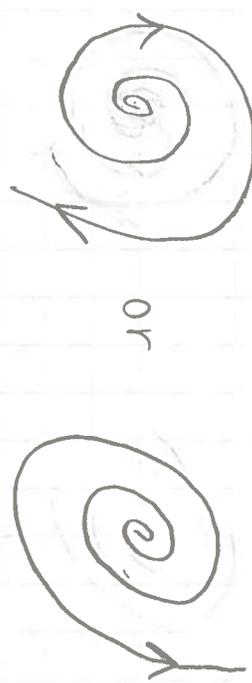
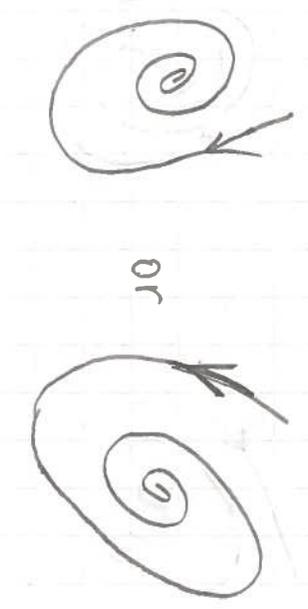
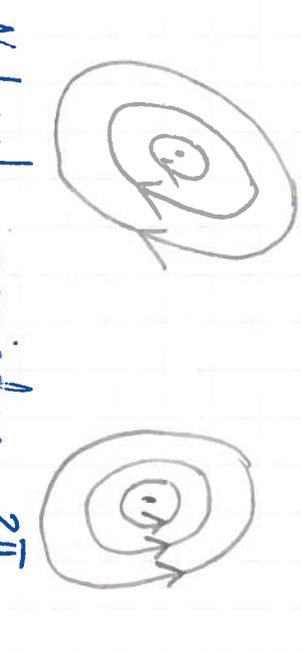
▲ So general solution to $\frac{dY}{dt} = \begin{pmatrix} -1 & 4 \\ -1 & -1 \end{pmatrix} Y$ in real form is

$$Y(t) = k_1 e^{-t} \begin{pmatrix} 2 \cos 2t \\ -\sin 2t \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} -2 \sin(2t) \\ -\cos(2t) \end{pmatrix}$$

If $Y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$,

so $k_1 = \frac{1}{2}$ and $k_2 = -1$.

③ Classification of equilibrium point (0,0) for the system
 $\frac{dY}{dt} = AY$, where matrix A has complex eigenvalues.

Eigenvalues	Name of eq. pt.	Stability	Pictures.
$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ α is positive	spiral source (or unstable focus)	unstable	
$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ α is negative	spiral sink (stable focus)	stable	
$\lambda_1 = i\beta$ $\lambda_2 = -i\beta$ (purely imaginary)	Center	stable	 <p>Natural period: $\frac{2\pi}{\beta}$ Natural frequency: $\beta/2\pi$</p>

Example 1.

The screenshot shows a software interface for solving differential equations. At the top, there is a vector field plot with axes from -3 to 3. A green curve is drawn through the field, starting at (1, 0) and moving towards the origin. Below the vector field are two time-series plots. The left plot shows a blue curve for y(t) starting at 0, dipping to a minimum of approximately -1.5, and then leveling off at 0. The right plot shows a red curve for x(t) starting at 0, dipping to a minimum of approximately -1.5, and then leveling off at 0. The interface includes several control panels:

- Equations:** $dx/dt = -x-y$ and $dy/dt = 2x-y$
- Initial Conditions:** $x = 2$, $y = 2$
- Range:** $dx/dt = -4$, $dy/dt = 2$
- Time Parameters:** $x_0 = 2$, $y_0 = 2$, $t_0 = 0$, $\Delta t = 0.05$
- Method:** Runge Kutta 4
- Options:** Draw Solutions, Draw Vectors
- Zoom:** Zoom In, Zoom Out, Reset

$$\begin{cases} \frac{dx}{dt} = -x-y \\ \frac{dy}{dt} = 2x-y \end{cases}$$

$$\lambda_1 = -1 + \sqrt{2}i$$

$\alpha = -1 \Rightarrow$ spiral sink
 $\beta = \sqrt{2} \Rightarrow$ natural period = $\frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$ counterclockwise rotation
 $F(x, y) = (-x-y, 2x-y)$
 $F(1, 0) = (-1, 2)$, $F(0, 1) = (-1, -1)$

5

Small α , large β

Example 2.

Clear Show Field

Clear Overlay Time Graphs

Clear Overlay Time Graphs

Runge Kutta 4 Draw Solutions Draw Vectors

$dx/dt = -0.5x + 9y$

$dy/dt = -x - 0.5y$

min x	-3	max x	3	x_0	2	delta t	0.05
min y	-3	max y	3	y_0	2		
min t	0	max t	25	t_0	0		
Reset Zoom Out Zoom In Solution Equations							

$\begin{cases} \frac{dx}{dt} = -0.5x + 9y \\ \frac{dy}{dt} = -x - 0.5y \end{cases}$
 $\lambda_1 = -0.5 + 3i$
 $\alpha = -0.5 \Rightarrow$ Spiral sink
 $\beta = 3 \Rightarrow$ natural period = $\frac{2\pi}{3}$
 $F(x,y) = (0.5x + 9y, -x - 0.5y)$
 $F(0,1) = (9, -0.5)$
 Clockwise rotation

Example 3.

Clear Hide Field

$x = 1$ $dx/dt = 2$

$y = 0$ $dy/dt = 2$

$dx/dt = 2*x - 6*y$

$dy/dt = 2*x + y$

min x	-2	max x	2
min y	-2	max y	2
min t	-3	max t	3

Reset Zoom Out Zoom In

x_0	1
y_0	0
t_0	0

Solution

delta t | 0.05

Equations

$$\frac{dx}{dt} = 2x - 6y$$

$$\frac{dy}{dt} = 2x + y$$

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{47}}{2} i$$

$\alpha = \frac{3}{2} \Rightarrow$ spiral source

$\beta = \frac{\sqrt{47}}{2} \Rightarrow$ natural period = $\frac{4\pi}{\sqrt{47}}$

$F(x, y) = (2x - 6y, 2x + y)$

$F(1, 0) = (2, 2)$ $F(0, 1) = (-6, 1)$

Rotation
counterclockwise

Example 4.

Clear Hide Field

$x = 0$
 $y = -1$

$dx/dt = 9$
 $dy/dt = 0$

Clear Overlay Time Graphs

Draw Solutions
 Draw Vectors

$dx/dt = -9y$
 $dy/dt = x$

min x	-4	max x	4	x_0	0
min y	-2	max y	2	y_0	-1
min t	0	max t	5	t_0	0

Reset Zoom Out Zoom In Solution Equations

$\Delta t = 0.05$

$$\begin{cases} \frac{dx}{dt} = -9y \\ \frac{dy}{dt} = x \end{cases}$$

$\lambda_1 = 3i$

$\alpha = 0 \Rightarrow$ center
 $\beta = 3 \Rightarrow$ natural period = $\frac{2\pi}{3}$
 $F(x, y) = (-9y, x)$, $F(1, 0) = (0, 1)$
 $F(0, 1) = (-9, 0)$

rotation
counterclockwise