Lecture on: Absolute value

. Please note that this material is not in the textbook.

- **Definition 1**: Given any real number x, its absolute value $|x|$ is defined by

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0, \\ -x, & \text{if } x < 0. \end{cases}$$

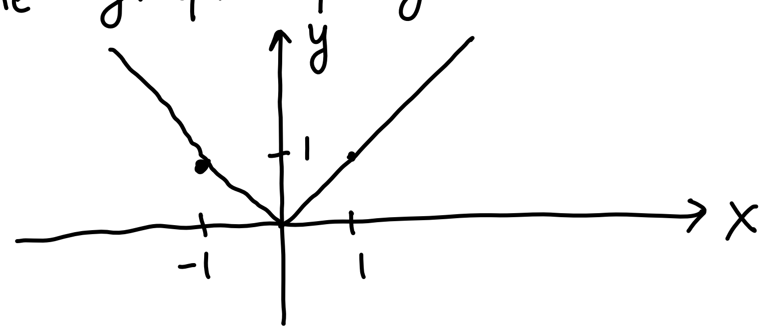
- **Def 2 (geometric)**: Given any real number x, $|x|$ is the distance from x to 0 on the number line.

- **Example 3**: $|5| = 5$, $|-3| = 3$,

$$|-x^2-5| = x^2+5,$$

$$|x^2-1| = \begin{cases} x^2-1, & \text{if } x^2-1 = (x-1)(x+1) \geqslant 0, \\ & \text{i.e } x \in (-\infty,-1] \cup [1,+\infty) \\ -x^2+1, & \text{if } x^2-1 < 0, \text{ i.e } x \in (-1,1) \end{cases}$$

- Here is the graph of $y = |x|$:



- **Proposition 4.** (Properties of absolute value)

Given arbitrary real numbers x, y,

1) $|x \cdot y| = |x| \cdot |y|$;

2) $x \leq |x|$;

3) (Triangle inequality) $|x+y| \le |x|+|y|$

• Exercise 1. Prove Proposition 4.

• Corollary 5 (Further properties of absolute value)

1) $|x| = |-x|$;
2) $|x_1 \cdot x_2 \cdot \ldots \cdot x_n| = |x_1| \cdot |x_2| \cdot \ldots \cdot |x_n|$;
3) $|x_1 + x_2 + \ldots + x_n| \le |x_1| + |x_2| + \ldots + |x_n|$;
4) $|x| - |y| \le |x+y|$

Proof: 1) $|-x| = |-1| \cdot |x|$ by prop. 4 and $|-1| = 1$ by def.

2) and 3): use induction and apply Prop. 4 properties 1 and 3.

4) We write $|x| = |(x+y) - y| \overset{\text{triangle ineq}}{\le} |x+y| + |-y| = |x+y| + |y|$. Thus $|x| - |y| \le |x+y|$.

▲

• Now let us work on some examples.

Example 6. Solve for x:

$$|x-1| + |x^2-1| + |x^3-1| = 0$$

Solution: Since for each a, $|a| \ge 0$, it must be true that $|x-1| = 0$, $|x^2-1| = 0$, and $|x^3-1| = 0$.

Thus $\boxed{x = 1}$ is the only solution

Example 7: Solve equation $|5-3x| = 3x-5$ for x.

Solution: Since $|5-3x| \ge 0 \Rightarrow 3x-5 \ge 0 \Leftrightarrow x \ge 5/3$.

On the other hand, if $x \geqslant 5/3$, then $5-3x \leq 0$ and $|5-3x| = -(5-3x) = 3x-5$.

$\boxed{\text{Answer. All } x \text{ satisfying } x \geqslant 5/3 \text{ or } [5/3, +\infty)}$

Example 8: Find all solutions of

$|x^2-6x+5| = |x^2-5|$.

Solution: Observe that if $|a| = |b|$, then either $a=b$ or $a=-b$.

Thus we have two cases.

Case 1: $x^2-6x+5 = x^2-5$

$\Downarrow$

$-6x = -10$

$\Downarrow$

$\boxed{x = 5/3}$ ← Answer →

Case 2: $x^2-6x+5 = -(x^2-5)$

$\Downarrow$

$2x^2-6x = 2x(x-3) = 0$

$\Downarrow$

$\boxed{x=0 \text{ or } x=3}$

Exercise 2: Find all solutions of the equations

a) $|x^2+3x| = |x^2+x+2|$;

b) $|x| - 2|x+3| = 2x$.

• Example 9: Solve the inequality $|x+3| \geqslant a$.

Solution: We need to consider two cases:

Case 1 $\boxed{\text{If } a \leq 0}$, then since $|x+3| \geqslant 0$ for all x, $|x+3| \geqslant a$ is true for all x. Thus if $a \leq 0$, then $-\infty < x < +\infty$ or $\boxed{x \in (-\infty, +\infty).}$

Case 2 If $a > 0$ and since $|x+3| = \begin{cases} x+3, & \text{if } x+3>0, \\ -(x+3), & \text{if } x+3<0 \end{cases}$

the inequality $|x+3| \geqslant a \iff x+3 \geqslant a$ or $-(x+3) \geqslant a$
$\iff x \geqslant -3+a$ or $(x+3 \leqslant -a$, i.e. $x \leqslant -3-a)$
In this case the answer is $x \in (-\infty, -a-3] \cup [a-3, +\infty)$.

$a>0$ $|x|>a \iff x>a$ or $x<-a$
$|x|<a$, $-a<x<a$

Note

• Example 10. Solve $|5-2x| + |x+7| - 3x = 6$.

Solution: First we rewrite

$$|5-2x| + |x+7| = 6+3x.$$

Since LHS of the equation is nonnegative, it should be true that $6+3x \geqslant 0$ or $x \geqslant -2$ and since $x \geqslant -2$, $x+7>0$ and $|x+7| = x-7$ and we can simplify:

$$|5-2x| + |x+7| = |5-2x| + x+7 = 6+3x$$

$$\Downarrow$$

$$|5-2x| = 2x-1.$$

Again there are two cases:

Case 1 $5-2x = 2x-1$
$4x = 6$, $\boxed{x = 3/2}$

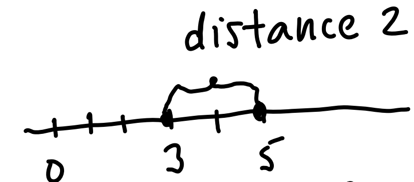
Case 2 $5-2x = -(2x-1)$
$5-2x = -2x+1$
has no solutions.

Geometric meaning of absolute value

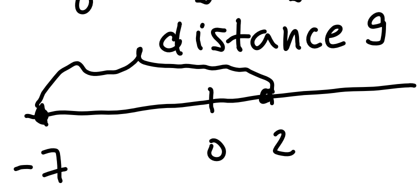
We will now discuss the geometric meaning of absolute value:

- <u>Observation 11</u>: $|x-a|$ = distance from x to a along a number line.

- <u>Example 12</u>: $|3-5|=2$



$|-7-2|=9$

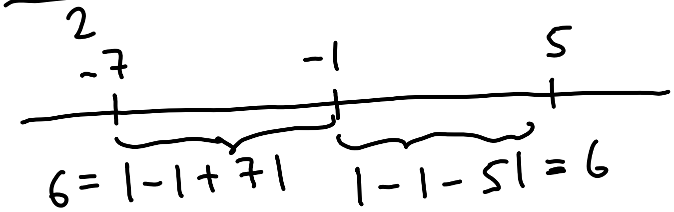


- <u>Example 13</u>: Solve geometrically:

a) $|x-5|=|x+7|$.

<u>Solution</u>: Since $|x-5|$ is the distance from x to 5 and $|x+7|$ is the distance from x to -7, the equation says that these two distances are equal, i.e. x is a midpoint between -7 and 5, i.e.

$x=\frac{-7+5}{2}=-1$. Here is the picture:



b) Solve $|x-5|<|x+7|$.

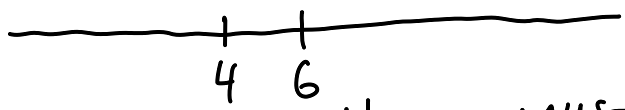
This inequality says that the distance from 5 to x is less than the distance from -7 to x. By looking at the picture above we see that x has to be on the right of -1. So $\boxed{x\in(-1,+\infty)}$.

• **Example 14.** Solve geometrically:

a) $|x-4|+|x-6|=10$.

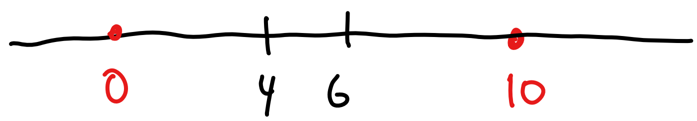
Solution: This equation says that the sum of distances from 4 and 6 to x is equal to 10.

We look at the picture:



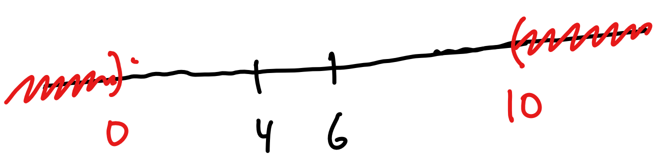
and see that there must be two solutions: one on the right of $[4,6]$ and another one on the left:

Answer: $x=0$ or $x=10$



b) $|x-4|+|x-6|>10$. Now we want the sum of distances from x to 4 and to 6 to be greater than 10:

Answer: $x\in(-\infty,0)\cup(10,+\infty)$

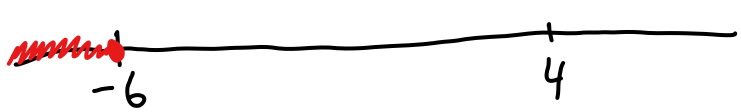


• **Example 15**: Solve geometrically $|x-4|-|x+6|=10$.

Solution: We rewrite $|x-4|=10+|x+6|$.

This equality says that the (distance from x to 4) is (the distance from x to -6) + 10.

We draw a picture:



Since the distance between -6 and 4 is 10, all points on the right of -6 will satisfy the equality in the example.

Answer: $x \in (-\infty, -6]$

- Exercise 3 How would you change an equation in example 15, so that the answer to the equation becomes $[4, +\infty)$?

- Exercise 4. Use the geometric meaning of the absolute value to solve the following equations and inequalities:

  a) $|x+2| \geq |x+9|$;

  b) $|x-5| + |x+7| = 12$;

  c) $|x-5| + |x+7| = 20$;

  d) $|x-4| - |x-9| = 6$;

  e) $|x-4| - |x-9| > 4$.

- Exercise 5: Graph the following functions (without calculator):

  a) $y = |x| + x$;

  b) $y = |2-x| - |2+x|$.

- Exercise 6: Solve the equation:

$$|x-2| + |x-1| + |x| + |x+1| + |x+2| = 6$$