03/24 Calc. lecture.

Lecture on: Sec. 5.2 Infinite series (part 2)
(Also Sec 9.2 from the handout)

Telescoping series.

Example 1. Discuss convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

We want to consider the sequence of partial sums:

$S_1 = \frac{1}{2}$, $S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3}$, $S_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4}$, ...

$$S_k = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{k(k+1)}$$

Formula for S_k?

$\frac{1}{n(n+1)} \leftarrow$ we think partial fractions

We write $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$, we want

$1 = A(n+1) + B(n)$

If $n = 0$, $1 = A(0+1) = A$, $\boxed{A = 1}$

$n = -1$, $1 = A\cdot 0 - B$, $\boxed{B = -1}$

So we have partial fraction $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

This means that $S_K = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{K(K+1)}$

$$S_K = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{K}} - \frac{1}{K+1}\right)$$

$$S_K = 1 - \frac{1}{K+1}.$$

Now $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{K\to\infty} S_K = \lim_{K\to\infty} \left(1 - \frac{1}{K+1}\right) = 1.$

So the telescoping series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

<u>Example 2.</u> $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}} \mathrel{.=} \sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

Formula for $S_K = \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{K}+\sqrt{K+1}}$

Algebra: $\frac{1}{\sqrt{n}+\sqrt{n+1}} = \left(\begin{array}{l}\text{multiply and}\\ \text{divide by conjugate}\end{array}\right) = \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n}+\sqrt{n+1})(\sqrt{n+1}-\sqrt{n})}$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1})^2 - (\sqrt{n})^2} = \frac{\sqrt{n+1}-\sqrt{n}}{n+1-n} = \sqrt{n+1} - \sqrt{n}.$$

$$S_K = (\cancel{\sqrt{2}} - \sqrt{1}) + (\cancel{\sqrt{3}} - \cancel{\sqrt{2}}) + (\cancel{\sqrt{4}} - \cancel{\sqrt{3}}) + \cdots + (\sqrt{K+1} - \cancel{\sqrt{K}}) = \sqrt{K+1} - 1$$

$\lim_{K\to\infty} S_K = +\infty$ (DNE)

Answer: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges to $+\infty$.

Example 3: $\displaystyle\sum_{n=2}^{\infty} \frac{5}{n^2-1}$

Algebra: $\dfrac{5}{n^2-1} = \dfrac{5}{(n-1)(n+1)} = \dfrac{A}{n-1} + \dfrac{B}{n+1}$

$5 = A(n+1) + B(n-1)$

$n = 1$: $5 = 2A$, $A = 5/2$

$n = -1$: $B = -5/2$

$$\frac{5}{n^2-1} = \frac{5}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right).$$

Computing partial sums:

$$S_K = \frac{5}{2}\left[\frac{1}{2^2-1} + \frac{1}{3^2-1} + \dots + \frac{1}{n^2-1}\right]$$

$$S_K = \frac{5}{2}\left[\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) + \dots + \left(\frac{1}{K-3} - \frac{1}{K-1}\right) + \left(\cancel{\frac{1}{K-2}} - \frac{1}{K}\right) + \left(\cancel{\frac{1}{K-1}} - \frac{1}{K+1}\right)\right]$$

$$S_K = 5/2\left[1 + 1/2 - \frac{1}{K} - \frac{1}{K+1}\right]$$

$$\lim_{K\to\infty} S_K = 5/2 \cdot 3/2 = 15/4.$$

Series converges to $15/4$ ← Answer

Divergence test.

Divergence test: If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Important! If $\lim_{n\to\infty} a_n = 0$, then divergence test is NOT APPLICABLE and you gain NO information about convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Explanation: Series $\sum_{n=1}^{\infty} a_n$ converges if and only if its series of partial sums $S_k = a_1 + a_2 + \ldots + a_k$ converges, to some number L. Observe that

$$S_k - S_{k-1} = (\cancel{a_1 + \ldots + a_{k-1}} + a_k) - (\cancel{a_1 + \ldots + a_{k-1}}) = a_k$$

and $\left(\sum_{n=1}^{\infty} a_n \text{ converges to } L\right) \Rightarrow$

$$\Rightarrow \left(\lim_{k\to\infty} a_k = \lim_{k\to\infty}(S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = L - L = 0\right)$$

So, if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{k\to\infty} a_k = 0$.

Another way to say this: if $\lim_{k\to\infty} a_k \neq 0$, then series $\sum_{n=1}^{\infty} a_n$ does not converge or it diverges.

Example: For each series below figure out if

- series converges
- series diverges
- we don't know yet (but might know later, when we learn more tests!)

a) $\sum_{n=1}^{\infty} n$ diverges $a_n = n$, $\lim_{n\to\infty} a_n = \infty \neq 0$ By div. test, diverges.

b) $\sum_{n=1}^{\infty} (-1)^n$ $a_n = (-1)^n$, $\lim_{n\to\infty} a_n = \text{DNE}$, series div. by div. test

c) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ← div. test is not applicable, need other tests.

d) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ $a_n = \sin\left(\frac{1}{n}\right)$ $\lim_{n\to\infty} \sin(1/n) = 0$ we don't know

e) $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ Geom: $1/5 + \frac{1}{25} + \dots$, $a = 1/5$, $r = 1/5$ Conv. to $\frac{a}{1-r} = \frac{1/5}{1-1/5}$

f) $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ ↖ diverges to $+\infty$ (telescoping)

g) $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ $a_n = \frac{n}{2n+1}$ $\lim_{n\to\infty} a_n = 1/2 \neq 0$ series div.

h) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \left(\ln(n) - \ln(n+1)\right)$ ← tescoping.

i) $\sum_{n=1}^{\infty} \frac{1}{n}$ can't use div. test, diverges (see last lecture)

Properties of infinite series.

If series $\sum_{n=1}^{\infty} a_n$ converges to a number A, i.e. $\sum_{n=1}^{\infty} a_n = A$

and if series $\sum_{n=1}^{\infty} b_n$ converges to a number B,

then 1) $\sum_{n=1}^{\infty} c \cdot a_n$ converges to $c \cdot A$

2) $\sum_{n=1}^{\infty} (a_n \pm b_n)$ converges to $A \pm B$.

Examples a) Observe that series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{2^2} + \cdots$ is geometric with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, it converges to $\frac{a}{1-r} = \frac{1/2}{1-1/2} = 1$. Similarly, series $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ converges to $\frac{1/5}{1-1/5} = 1/4$.

Then, $\sum_{n=1}^{\infty} \left[3\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{5}\right)^n\right] = 3\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n - 4\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$

$= 3 \cdot 1 - 4 \cdot \frac{1}{4} = -1$

b) $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) \neq \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1}$, since series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n+1}$ both diverge!