Lecture on: Section 5.2. Infinite series

- Infinite series is a sum of infinitely many terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

terms of the series

- The terms of the series are numbers, so we are adding together infinitely many numbers. The resulting sum can be a specific finite number, $+\infty$, $-\infty$, or the sum is undefined.

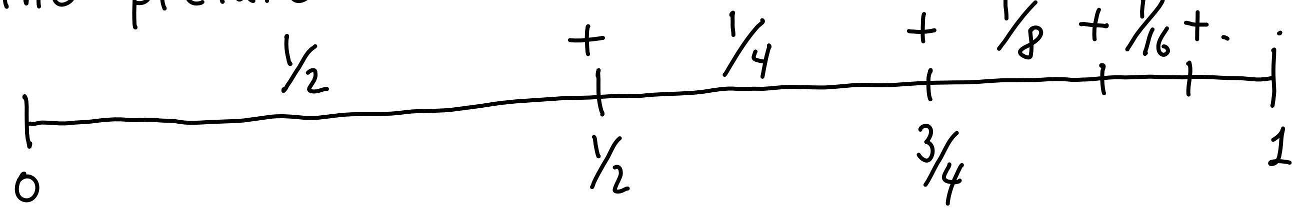
Example 1 (Geometric series):

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \dots$$

This series is called geometric because we are adding terms that form geometric progression.

What is the sum of this series?

One way to find this sum is to look at the picture below



From the picture we can see that

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \dots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$

(The sum of the lengths of intervals is 1).

- How could we find the same sum algebraically?

We will consider the sequence of associated partial sums:

$$S_1 = \frac{1}{2}, \quad S_2 = \frac{1}{2} + \frac{1}{2^2} = 3/4, \quad S_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{7}{8}$$

$$S_4 = \frac{15}{16}, \quad S_k = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k}.$$

In general the k^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ is the number $S_k = a_1 + a_2 + \dots + a_k$.

Thus each series has the corresponding sequence of partial sums: $S_1, S_2, \dots, S_k, \dots$

By definition, $\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k$.

For the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, the sequence of partial sums is $\{S_k\} = \left\{1 - \frac{1}{2^k}\right\}$ and thus

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{k\to\infty}\left(1 - \frac{1}{2^k}\right) = 1.$$

So we say that the sum of this series is 1 or that the series converges to 1

So, to summarize

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k \text{ , where } S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \dots + a_k$$

The sum of the series is the limit of a sequence of partial sums. The series converges to a number S if $\lim_{k\to\infty} S_k = S$, and diverges if $\lim_{k\to\infty} S_n$ does not exist.

General Geometric series.

A general geometric series is the series of the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots$

a) The series from example 1 was

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

We see that $a = \frac{1}{2}$ and $r = \frac{1}{2}$.

b) The series $\sum_{n=1}^{\infty} \frac{2}{3^n}$ starts as $2/3 + \frac{2}{9} + \frac{2}{3^3} + \dots$;

So in this case $a = 2/3$ and $r = \frac{1}{3}$.

c) The series $\sum_{n=1}^{\infty} 2(-1)^n = -2+2-2+2-2+\dots$

has $a = -2$ and $r = -1$.

- Let us figure out the sequence of partial sums for $\sum_{n=1}^{\infty} ar^{n-1}$:

$$S_k = \sum_{n=1}^{k} ar^{n-1} = a + ar + ar^2 + \dots ar^{k-1}$$

There is a neat trick to find a formula for for this sum: assume $r \neq 1$ and consider

$$S_k - r \cdot S_k = (a + ar + \dots + ar^{k-1}) - r \cdot (a + ar + \dots + ar^{k-1})$$

$$= (a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{k-1}}) - (\cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{k-1}} + ar^k)$$

$$= a - ar^k.$$

Thus $S_k(1-r) = a(1-r^k)$ and

$$S_k = a + ar + \dots + ar^{k-1} = \begin{cases} \dfrac{a(1-r^k)}{1-r}, & \text{when } r \neq 1 \\ a + a + \dots + a = ak, & \text{when } r = 1. \end{cases}$$

Now observe that if $a \neq 0$,

$$\lim_{k \to \infty} r^k = \begin{cases} 0, & \text{if } -1 < r < 1, \text{ or } |r| < 1 \\ 1, & \text{if } r = 1 \\ \text{does not exist (DNE)}, & \text{if } |r| > 1, \text{ or } r = -1 \end{cases}$$

(Note: When $r > 1$, $\lim_{k \to \infty} r^k = +\infty$)

Thus, if $r \neq 1$, $a \neq 0$

$$\lim_{k \to \infty} S_k = \lim_{k \to \infty} \left(\frac{a(1-r^k)}{1-r} \right) = \begin{cases} \dfrac{a}{1-r}, & |r| < 1, \\ \text{DNE}, & |r| > 1, \text{ or } r = -1. \end{cases}$$

And if $r = 1$,

$$\lim_{k \to \infty} S_k = \lim_{k \to \infty} ak = +\infty \text{ or } -\infty \text{ (i.e. DNE)}.$$

To summarize:

For geometric series $\sum_{n=1}^{\infty} ar^{n-1}$, $a \neq 0$,

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{k \to \infty} S_k = \begin{cases} \frac{a}{1-r}, & |r|<1, \text{ series converges} \\ DNE, & \text{series diverges if } |r| \geq 1 \end{cases}$$

Example 2. a) $\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 2 + \frac{2}{3} + \frac{2}{9} + \dots$

$= a + ar + ar^2 + \dots$

Thus, this is geometric series with $a = 2$ and $r = 1/3$. Since $|r| = 1/3 < 1$, by the formula above

$$\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = \frac{2}{1 - 1/3} = 3,$$ so one can say

that $\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3$ or $\boxed{\text{the series converges to } 3}$.

Example 3: Consider series $2/5 - \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 - \dots$

Comparing this series to $a + ar + ar^2 + \dots$ we see that $a = 2/5$, $r = -2/5$. Since $|r| = 2/5 < 1$,

$$2/5 - (2/5)^2 + (2/5)^3 - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot (2/5)^n$$

$$= \frac{2/5}{1-(-2/5)} = \frac{2/5}{7/5} = 2/7.$$ $\boxed{\text{So the series converges to } 2/7}$.

Example 4. $\sum_{n=1}^{\infty} (4/3)^n = 4/3 + (4/3)^2 + \dots$

$a = 4/3$, $r = 4/3$, $|r| = 4/3 > 1$, $\boxed{\text{so series diverges}}$. Note, in fact, $\sum_{n=1}^{\infty} (4/3)^n = +\infty$,

so series diverges to $+\infty$.