Lecture: 04/27/2020

#1a (hwk 23)

$A_1 = \{a_{11}, a_{12}, \ldots, a_{1n}, \ldots\}$

$A_2 = \{a_{21}, a_{22}, \ldots, a_{2n}, \ldots\}$

$\vdots$

$A_n = \{a_{n1}, a_{n2}, \ldots, a_{nn}, \ldots\}$

$\{a_{11}, a_{21}, \ldots, a_{n1}, a_{12}, a_{22}, \ldots, a_{n2}, \ldots\}$

b) Why do we still have an infinite list?

We can organize removal so that elements of A_1 are never removed and A_1 was infinite, so the list for $\bigcup_{i=1}^{n} A_i$ is still infinite.

$A_1 \subseteq \bigcup_{i=1}^{n} A_i$

Why Cantor diagonal method does not prove that $\mathbb{Q}$ is uncountable?

"Proof that $\mathbb{Q} \cap (0,1)$ are uncountable".

$q_1 = .\, q_{11} q_{12} \ldots q_{1m} \ldots$

$\vdots$

$q_n = .\, q_{n1} q_{n2} \ldots q_{nm} \ldots$

Let $b = .b_1 b_2 \dots b_n \dots$ Need b to be rational. can't be proved

$$b_m = \begin{cases} 3, & \text{if } q_{mm} \neq 3 \\ 4, & \text{if } q_{mm} = 3 \end{cases}$$

$\Rightarrow$ $\cancel{\mathbb{Q} \cap (0,1)}$ is uncountable? False

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Similarly this prove does not work well to show that irrational numbers are uncountable $\leftarrow$ a true fact.

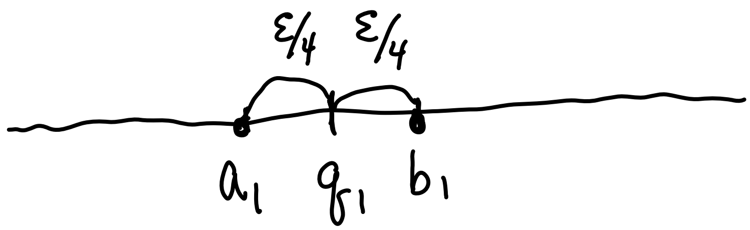
We will show that $\mathbb{Q}$ has measure zero

<u>Step 1.</u> $\mathbb{Q}$ is countable, so we can list all rational numbers: $q_1, q_2, \dots, q_n, \dots$

<u>Step 2</u> Choose any $\varepsilon > 0$

Cover q_1 by an interval $I_1 = [a_1, b_1]$ such that

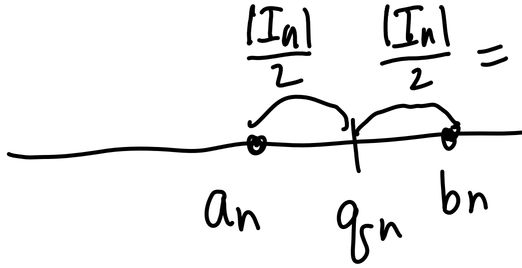
$$|I_1| = b_1 - a_1 = \frac{\varepsilon}{2}$$



Cover q_2 by I_2 of length $\frac{\varepsilon}{2^2}$

$\vdots$

Cover q_n by I_n of length $\frac{\varepsilon}{2^{n+1}}$



$$\frac{|I_n|}{2} = \frac{\varepsilon}{2^{n+2}}$$

$$a_n = q_n - \frac{|I_n|}{2}$$
$$b_n = q_n + \frac{|I_n|}{2}$$
$$\boxed{b_n - a_n = |I_n|}$$

$\vdots$

Each rational number is covered by at least one interval

$$\mathbb{Q} \subseteq \bigcup_{n=1}^{\infty} I_n$$

Sum of lengths of intervals

$$\sum_{n=1}^{\infty} |I_n| = \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \dots + \frac{\varepsilon}{2^{n+1}} + \dots$$

$$= a + ar + ar^2 + \dots = \frac{a}{1-r} =$$

$a = \varepsilon/2$

$r = 1/2$

$$= \frac{\varepsilon/2}{1 - 1/2} = \varepsilon.$$

$\text{length}(\mathbb{Q}) \leq \varepsilon$ for all $\varepsilon \Rightarrow \text{length}(\mathbb{Q}) = 0$.

$\{1, 2, 3\}$ $\qquad\qquad$ $\{1, 2, 3, 4, \dots\}$

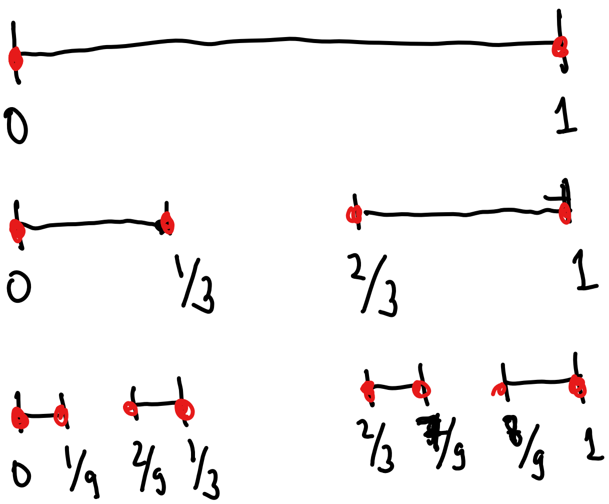
$0\,1\,0 \mapsto \{2\}$ $\qquad\qquad$ $0\,1\,0\,0\,1 \dots$

$1\,1\,0 \mapsto \{1, 2\}$ $\qquad\qquad$ $\{2, 5, \dots\}$

Uncountable set of length zero?

Yes!

Cantor set.



$C_0, \quad \ell(C_0) = 1$

$C_1, \quad \ell(C_1) = 2/3$

$C_2, \quad \ell(C_2) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

$C_n, \quad \ell(C_n) = (2/3)^n$

2^n pieces of length $(1/3)^n$.

$$C = \bigcap_{n=0}^{\infty} C_n.$$

$C \subset C_n$ for all n.

(contains endpoints)

C is infinite.

length of C is

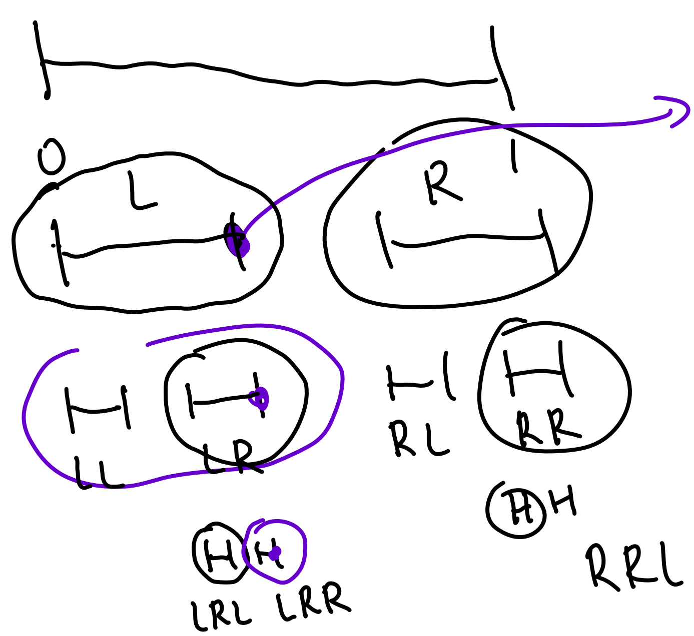
$C \subset C_n, \quad \ell(C) < \ell(C_n) = (2/3)^n$

$\ell(C) < (2/3)^n$

$$\Rightarrow \boxed{\ell(C) \le \lim_{n\to\infty} \left(2/3\right)^n = 0}$$

Is there anything else in C besides endpoints?

0100100001 . . .
LRLLRLLLR . . . . . $\leftarrow$ address in C.



LRRRRRR . . .

LRLLRLLLLL . .

Nested sequence of closed intervals

L
LR
LRL
LRLL
$\vdots$

$\{\text{Point}\} = \bigcap(\text{all these intervals})$
$= \emptyset$