**Lecture:** 04/27 Calculus lecture

Solutions to Quiz #26.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

(a) $\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \dots$

Since $\frac{1}{5040} < .001$, $\sin 1 \approx 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120} \approx .841$

(b) $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{7!} + \dots$

Thus, $\lim\limits_{x \to 0} \dfrac{\sin(x^2) - x^2 + x^6/6}{x^{10}} = \lim\limits_{x \to 0} \dfrac{\frac{x^{10}}{120} - \frac{x^{14}}{7!} + \dots}{x^{10}}$

$= \lim\limits_{x \to 0} \left( \frac{1}{120} - \frac{x^4}{7!} + \dots \right) = \frac{1}{120}.$

(c) $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx = \int_0^1 \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x}\, dx$

$\displaystyle= \int_0^1 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\, dx = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \dots \Big|_0^1$

$\displaystyle= 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$

(d) $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx \approx 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{35280} + \dots$

Since $\frac{1}{35280} < \frac{1}{10000} = .0001 \Rightarrow \displaystyle\int_0^1 \frac{\sin x}{x}\, dx \approx 1 - \frac{1}{18} + \frac{1}{600}$

$= \frac{1703}{1800} \approx .9461$

# Homework questions.

#25 on p. 644

$f(x) = \frac{2}{x}$, $P_3(x)$, $c = 1$.

$$P_3(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3$$

$f(x) = \frac{2}{x}$, $f(1) = 2$

$f'(x) = -\frac{2}{x^2}$, $f'(1) = -2$

$f''(x) = +\frac{4}{x^3}$, $f''(1) = 4$

$f^{(3)}(x) = -\frac{12}{x^4}$, $f^{(3)}(1) = -12$

$$P_3(x) = 2 + (-2)\cdot(x-1) + \frac{4}{2!}(x-1)^2 + \frac{-12}{3!}(x-1)^3$$

$$= \underbrace{2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3}$$

## 2nd way. Use power series expansions:

Will expand $f(x) = \frac{2}{x}$ as a power series centered at $x = 1$:

$$\boxed{\frac{2}{x} = \frac{2}{1 + x - 1} = \frac{a}{1-r}}, \quad a = 2, \ r = -(x-1)$$

$$\frac{a}{1-r} = a \cdot \sum_{n=0}^{\infty} r^n = 2 \sum_{n=0}^{\infty} \left(-(x-1)\right)^n$$

$|r| < 1$

$= |x-1| < 1$

$\boxed{0 < x < 2}$

$$= \underbrace{2 \cdot \sum_{n=0}^{\infty} (-1)^n (x-1)^n}$$

Taylor series for $f(x) = \frac{2}{x}$ centered at $x = 1$

$$= \boxed{2\left(1 - (x-1) + (x-1)^2 - (x-1)^3\right.} \left. + \dots\right)$$

$P_3(x)$

#41 $P_4(x)$ of $y = e^{4x}$ to compute

$f(\frac{1}{4}) = ?$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{4x} = \underbrace{1 + 4x + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \frac{(4x)^4}{4!}}_{P_4(x)} + \cdots$$

$$f(1/4) \approx P_4(1/4) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$$

$$\overset{\shortparallel}{e^{4 \cdot 1/4}} = \frac{48 + 12 + 4 + 1}{24} = \frac{65}{24} \approx 2.708 \approx 2.71$$

$$\overset{\shortparallel}{e} = 2.71$$

#20 How to compute $P_4(x)$ for $f(x) = x^2 e^{-x}$ quickly.

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$$

$$x^2 e^{-x} = x^2 \cdot \left(1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots\right)$$
$$= x^2 - x^3 + \frac{x^4}{2} - \dots$$

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$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$$

$$\boxed{P_4(x) = x^2 - x^3 + \frac{x^4}{2}}$$

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Limit comparison always works. Sometimes direct comparison gives an answer faster.

Ex: $\sum_{n=1}^{\infty} \frac{5^n - n^2}{7^n + n^2}$

Limit comparison, compare with $\sum_{n=1}^{\infty} \frac{5^n}{7^n}$

$$\lim_{n\to\infty} \frac{\frac{5^n - n^2}{7^n + n^2}}{5^n/7^n} = \dots$$

- Direct comparison

$$\underbrace{\sum_{n=1}^{\infty} \frac{5^n - n^2}{7^n + n^2}}_{\text{converges.}} \le \sum_{n=1}^{\infty} \frac{5^n}{7^n} \leftarrow \text{geometric} \quad r = \frac{5}{7} < 1$$

converges.

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Ex: $\sum_{n=1}^{\infty} \frac{n^2+3n+1}{n^4-n+5}$

Easy to use limit comparison test:

$$\sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2} \leftarrow \text{converges.}$$

$$\lim_{n \to \infty} \frac{n^2+3n+1}{n^4-n+5} \Big/ \frac{1}{n^2} = 1.$$

$\sum \frac{3^n}{n!} \leftarrow$ Ratio test.

$\sum \left(\frac{2}{3}\right)^n \leftarrow$ geom. series.

$\sum (-1)^n \underbrace{\sin(1/n)}_{\downarrow 0 \text{ as } n \to \infty} \leftarrow$ converges.

$$\sum (-1)^n \cos(1/n) \leftarrow \text{diverges}$$

$$\downarrow n \to \infty$$

$$1$$

$$\lim_{n \to \infty} (-1)^n \cos(1/n) \quad \text{does not exist}$$

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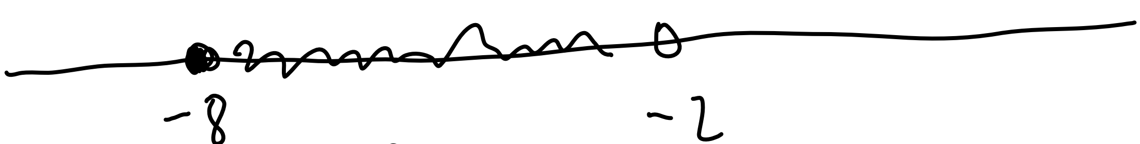
<u>Example</u>: $\displaystyle\sum_{n=0}^{\infty} \frac{(x+5)^n}{3^n \cdot n} = f(x)$

Converges: $\left|\frac{x+5}{3}\right| < 1$, $|x+5| < 3$

$-3 < x+5 < 3$, $\boxed{-8 < x < -2}$




$x = -2$ $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ diverges

$x = -8$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converges

Interval of conv. $= [-8, -2)$

$$f'(x) = \sum_{n=1}^{\infty} \frac{(x+5)^{n-1}}{3^n}$$

$(-8, -2)$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{(x+5)^{n+1}}{(n+1)n \cdot 3^n}$$

$[-8, -3]$