Lecture: 04/24 Proof lecture

Set $S$ is countable : there is bijection $f: \mathbb{N} \to S$

Set $S$ is not countable : there is no bijection from $\mathbb{N} \to S$

Set $S$ is uncountable : set is infinite and there is no bijection from $\mathbb{N} \to S$.

$\mathbb{Z}$ is countable

$\{1, 2, 3\}$ is not countable

$\mathbb{R}$ is both not countable and uncountable

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Definition of inverse function:

$f: S \to T$ a bijection, then it has inverse $g: T \to S$ defined by

$\forall t \in T \quad g(t) = s_0$ is a unique solution $s_0$ of $f(s) = t$

$g = f^{-1}$ $\quad f \circ f^{-1}(t) = t \quad f^{-1} \circ f(s) = s$

$f \circ g(t) \overset{?}{=} t \quad g \circ f(s) = s.$

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$A_1, A_2, \ldots, A_n, \ldots$ ← countable

$A_1 = \{a_{11}, a_{12}, \ldots, a_{1n}, \ldots\}$

$A_2 = \ldots$

$A_n = \ldots$

Why can you do it.

How to make combined list:
should have no repetitions and it should be infinite.

Then there is a bijection $f: \mathbb{N} \to \{\text{List}\}$

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Homework questions.

4. Suppose set of all infinite sequences of 0's and 1's is countable, then

$S = \{a_1, a_2, \ldots, a_n, \ldots\}$ ↙ list of all such sequences.

$a_1 = a_{11} a_{12} \ldots a_{1n} \ldots$ 1010000

$a_2 =$ $a_{11}=1, a_{12}=0, a_{13}=1$

$a_{ij} = 0$ or $1$.

$\vdots$

$a_n = a_{n1} a_{n2} \ldots a_{nn} \ldots$

$\vdots$

Find a sequence $b = b_1 b_2 \ldots$

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Sec 21 (finished)

Hierarhy of infinities.

Given two sets A and B

- If there is a bijection $f: A \to B \Rightarrow$ $\text{card}(A) = \text{card}(B)$
- if there is an injection $g: A \to B$, then $\text{card}(A) \leq \text{card}(B)$
- if there is a surjection $h: A \to B$,

then $card(A) \geq card(B)$.

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## Schröder – Bernstein theorem:

If there is an injection $g: A \to B$ and a surjection $h: A \to B$, then there is a bijection $f: A \to B$.

( $card(A) \leq card(B)$ and $card(A) \geq card(B) \Rightarrow card(A) = card(B)$ )

Ex: Prove that $(0,1)$ and $[0,1]$ have the same cardinality.

1st way: Find a bijection $f: (0,1) \to [0,1]$.

2nd way: Applying Schröder – Bernstein theorem

Need injection $g: (0,1) \to [0,1]$

$g(x) = x$

Need a surjection $h: (0,1) \to [0,1]$.

$$h(x) = 2x - \tfrac{1}{2} : (0,1) \to (-\tfrac{1}{2}, \tfrac{3}{2})$$

$\Rightarrow$ there is a bijection $f: (0,1) \to [0,1]$

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<u>Def:</u> If $S$ is a set, let $P(S)$ be the set consisting of all the subsets of $S$.

<u>Ex:</u> $S = \{1, 2\}$, $P(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

We proved before that if $S = \{1, 2, \ldots, n\}$, then $P(S)$ has $2^n$ elements.

$$\boxed{\operatorname{card}(P(S)) = 2^{\operatorname{card}(S)}}$$

<u>Proposition 21.5</u> Let $S$ be a set, then there is no bijection from $S$ to $P(S)$.

Comment: $g: S \to P(S)$ injection

$g(s) = \{s\}$, so

for free

$$\boxed{\begin{array}{l} \text{card}(S) \le \text{card}(P(S)) \\ \text{also } \text{card}(S) \ne \text{card}(P(S)) \end{array}}$$

will prove

$\Rightarrow \quad \text{card}(S) < \text{card}(P(S))$

$\mathbb{N}$, $\text{card}(\mathbb{N}) = \aleph_0$

$\text{card}(P(\mathbb{N})) = \text{card}(\mathbb{R}) = \underset{\aleph_1}{\underbrace{2^{\aleph_0}}} > \aleph_0$

$\text{card}(P(\mathbb{R})) = \underbrace{2^{\aleph_1}}_{\aleph_2} > \aleph_1$

$\vdots$

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Proof: By contradiction.

Suppose there is a bijection

$f: S \to P(S)$

For each $s \in S$, let us denote

$f(s) = A_s \leftarrow$ subset in $S$

(For instance: $S = \mathbb{N}$, then
$f : \mathbb{N} \to P(\mathbb{N})$, then
$f(1) = A_1$, $f(2) = A_2$, ...
$A_1 = \{1, 2, 3\}$, $A_2 = \{2, 4, 6, 8, \ldots\}$,
$A_3 = \{4, 5, 7\}$ )

Now define the set

$$B = \{ s \in S, \text{ such that } s \notin A_s \}$$

(In our example: $1 \in A_1 \Rightarrow 1 \notin B$
$2 \in A_2 \Rightarrow 2 \notin B$,
$3 \notin A_3 \Rightarrow 3 \in B$ ).

Since we assumed that $f : S \to P(S)$ is a bijection and $B \in P(S) \Rightarrow$
$\exists\, \tilde{s} \in S$ such that $f(\tilde{s}) = B = A_{\tilde{s}}$

Question: is $\tilde{s} \in B$ or not?

- Suppose $\tilde{s} \in B \iff \tilde{s} \in A_{\tilde{s}} \Rightarrow \tilde{s} \notin B$.
(this can't happen).
- Suppose $\tilde{s} \notin B \iff \tilde{s} \notin A_{\tilde{s}} \Rightarrow \tilde{s} \in B$
(this can't happen too).

This is a contradiction with the existence of $\tilde{s}$ s.t. $f(\tilde{s}) = B$, so $f$ can't be onto $\Rightarrow$ $f$ is not a bijection!