Lecture: 04/24 Calculus lecture

Class announcements

1. Bonus quiz and the last homework are due on Monday.
2. Review sheet (with answers) is available on Calculus II web site.
3. There will be test 3 (same format as test 2) on Tuesday. We will review for test 3 on Monday.
4. If there is interest, I can hold an additional optional review session on Monday evening.

Quiz # 25 (solutions)

(a)

$f(x) = \sin x$ $\qquad f(\pi/6) = \frac{1}{2}$

$f'(x) = \cos x$ $\qquad f'(\pi/6) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x$ $\qquad f''(\pi/6) = -\frac{1}{2}$

$f'''(x) = -\cos x$ $\qquad f'''(\pi/6) = -\sqrt{3}/2$

$f^{(4)}(x) = \sin x$ $\qquad f^{(4)}(\pi/6) = 1/2$

$\ldots \qquad \ldots$

$$\boxed{f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots}$$

Thus $\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) - \frac{1}{2\cdot 2!}(x-\pi/6)^2 - \frac{\sqrt{3}}{2\cdot 3!}(x-\pi/6)^3 + \frac{1}{2\cdot 4!}(x-\pi/6)^4 + \ldots$

$-\infty < x < \infty$

b)

$g(x) = \ln x$ $\qquad g(2) = \ln 2$

$g'(x) = \frac{1}{x}$ $\qquad g'(2) = \frac{1}{2}$

$g''(x) = -\frac{1}{x^2}$ $\qquad g''(2) = -\frac{1}{2^2}$

$g^{(3)}(x) = \frac{2}{x^3}$ $\qquad g^{(3)}(2) = \frac{2}{2^3}$

$$g^{(4)}(x) = -\frac{2\cdot 3}{x^4} \qquad g^{4}(2) = -\frac{2\cdot 3}{2^4}$$

$$\vdots$$

$$g^{(n)}(x) = \frac{(-1)^{n+1}\cdot(n-1)!}{x^n} \; ; \; = \frac{(-1)^{n+1}(n-1)!}{2^n}$$

Thus $$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!\,2^n}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n\cdot 2^n}(x-2)^n$$

$$0 < x \le 4$$

Homework questions.

Note: Maclaurin series for $f(x) = (1+x)^k$ is

$f'(x) = k(1+x)^{k-1}$

$f''(x) = k(k-1)(1+x)^{k-2}$

$\vdots$

$f^{(n)}(x) = k(k-1)\ldots(k-(n-1))(1+x)^{k-n}$

$f(0) = 1$

$f'(0) = k$

$f''(0) = k(k-1)$

$\vdots$

$f^{(n)}(0) = k(k-1)\ldots(k-(n-1))$

So $$(1+x)^k = \underbrace{1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots + \frac{k(k-1)\ldots(k-(n-1))}{n!}x^n + \ldots}_{\text{binomial series}}$$

binomial series (see also example 5 on p. 669)

17 $$\frac{1}{(1+x)^2} = (1+x)^{-2}, \quad \boxed{k = -2}$$

$$(1+x)^{-2} = 1 - 2x + \frac{-2(-2-1)}{2!}x^2 + \frac{(-2)(2-1)(-2-2)}{3!}x^3 + \ldots + \frac{-2(-2-1)\ldots(-2-(n-1))}{n!}x^n + \ldots = 1 - 2x + x^2 - \frac{8}{3!}x^3 + \ldots \frac{(-1)^n\, 2\cdot 3\cdot 4\cdots(n+1}{n!}$$

Sec. 9.7. Taylor Polynomials and Approximations x^n

Recall Taylor series for $f(x)$ centered at $x=c$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = \underbrace{f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\dots+\frac{f^{(n)}(c)}{n!}(x-c)^n}_{P_n(x)}+\dots$$

Taylor polynomial of degree m centered at $x=c$ for $f(x)$ is a polynomial $P_m(x)$ given by

$$P_m(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\dots+\frac{f^{(m)}(c)}{m!}(x-c)^m$$

($P_m(x)$ = polynomial of degree m is m^{th} partial sum of Taylor series).

Maclaurin polynomial = Taylor polynomial centered at $c=0$

Examples:

a) Since $e^x = 1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+\dots$,

its m^{th} Maclaurin polynomial is

$$P_m(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^m}{m!}$$

In particular,

$$P_0(x) = 1$$

$$P_1(x) = 1+x$$

$$P_2(x) = 1+x+\frac{x^2}{2!}$$

$$P_3(x) = 1+x+\frac{x^2}{2}+\frac{x^3}{6}$$

b) Since $\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 2^n} (x-2)^n$ on $(0, 4]$

its 2^{nd} Taylor polynomial centered at $x=2$

$$P_m(x) = \ln 2 + \sum_{n=1}^{m} \frac{(-1)^{n+1}}{n \cdot 2^n}(x-2)^n$$

is $P_0(x) = \ln 2$

$$P_1(x) = \ln 2 + \frac{x-2}{2}$$

$$P_2(x) = \ln 2 + \frac{x-2}{2} - \frac{1}{2 \cdot 2^2}(x-2)^2$$

c) Use the definition to find 2^{nd} Taylor polynomial for $f(x) = \sqrt{x}$ centered at $c = 4$.

$$\boxed{P_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2}$$

$$f(x) = \sqrt{x} = x^{1/2}$$

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-3/2}$$

$$f(4) = 4^{1/2} = 2$$

$$f'(4) = \frac{1}{2} 4^{-1/2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$f''(4) = -1/4 \, 4^{-3/2} = -\frac{1}{4} \cdot \frac{1}{4^{3/2}} = -\frac{1}{4} \cdot \frac{1}{8} = -1/32$$

$$\boxed{P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2}$$

$\sqrt{x} \approx P_2(x)$ near $x = 4$

$$\sqrt{4.5} \approx P_2(4.5) = 2 + \frac{1}{4}\underbrace{(4.5-4)}_{1/2} - \frac{1}{64}\underbrace{(4.5-4)^2}_{1/2^2}$$

$$= 2 + \frac{1}{8} - \frac{1}{256} = \frac{512 + 32 - 1}{256}$$

$\parallel$

2.12132

$$= \frac{543}{256} \approx 2.12093$$

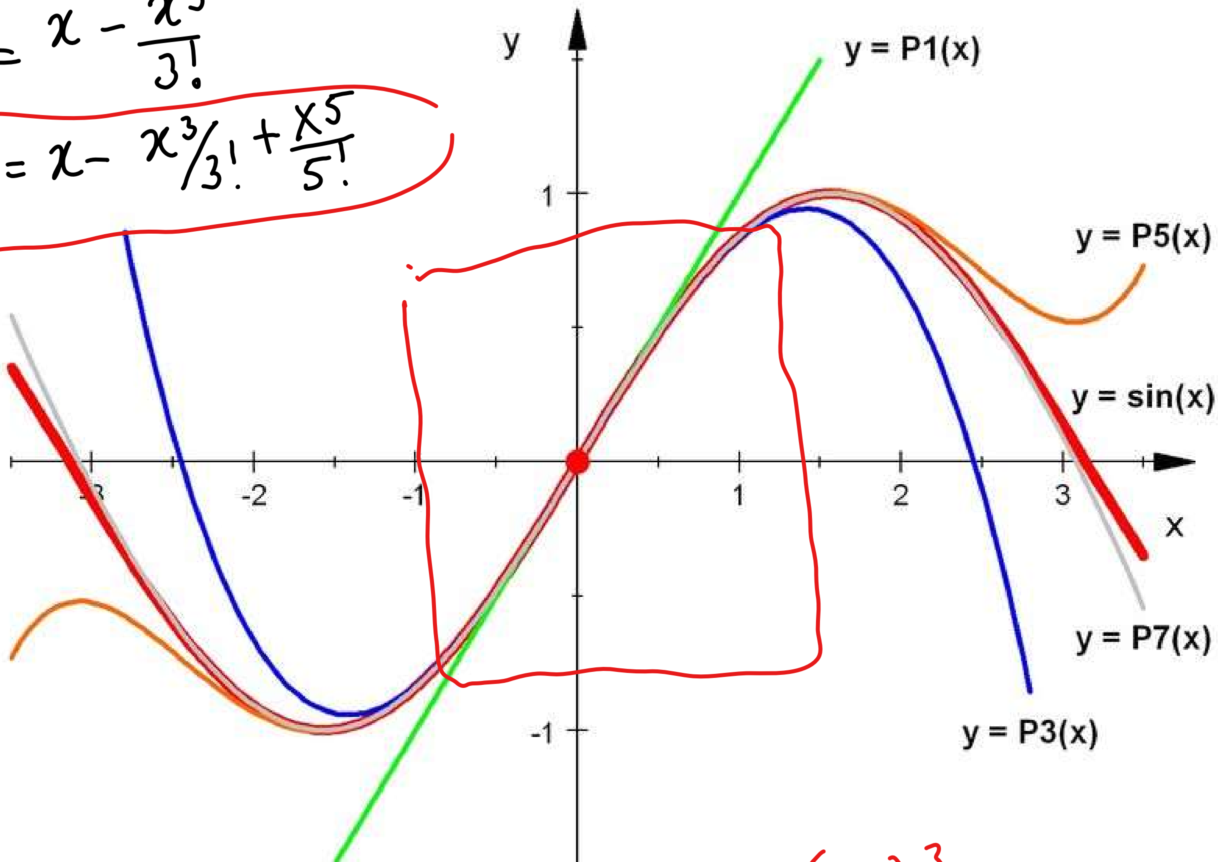
Approximating $y = f(x) = \sin x$ by its Maclaurin polynomials.

$$\sin x = x - \frac{x^3}{3!} + x^5/5! - \cdots$$

$$P_1(x) = x$$

$$P_3(x) = x - \frac{x^3}{3!}$$

$$P_5(x) = x - x^3/3! + \frac{x^5}{5!}$$



$$\sin(1/2) \approx P_3(1/2) \simeq \frac{1}{2} - \frac{(1/2)^3}{6} = 1/2 - \frac{1}{48}$$

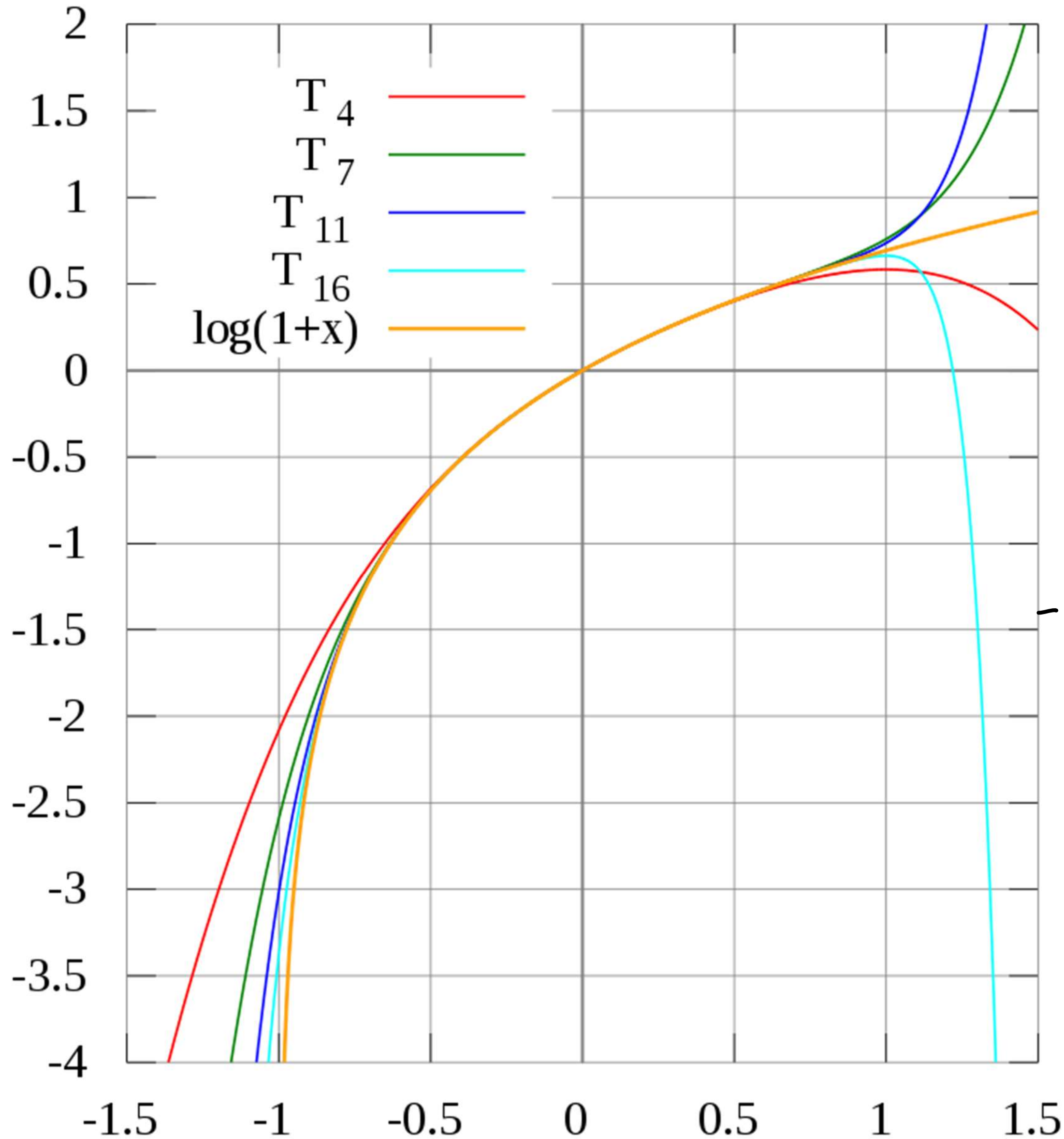
$$= \frac{24-1}{48}$$

$$= 23/48 \approx .47917$$

$\sin(1/2)$ = .47942 numerically

$1/2$: $\frac{180}{2\pi}$ degrees.

Approximating $y = f(x) = \ln(1+x)$ by its Maclaurin polynomials $T_4(x) = P_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$



$-1 < x \leq 1$

Function $f(x)$ whose Maclaurin series is not equal to $f(x)$ anywhere except at $x=0$:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = \ldots$

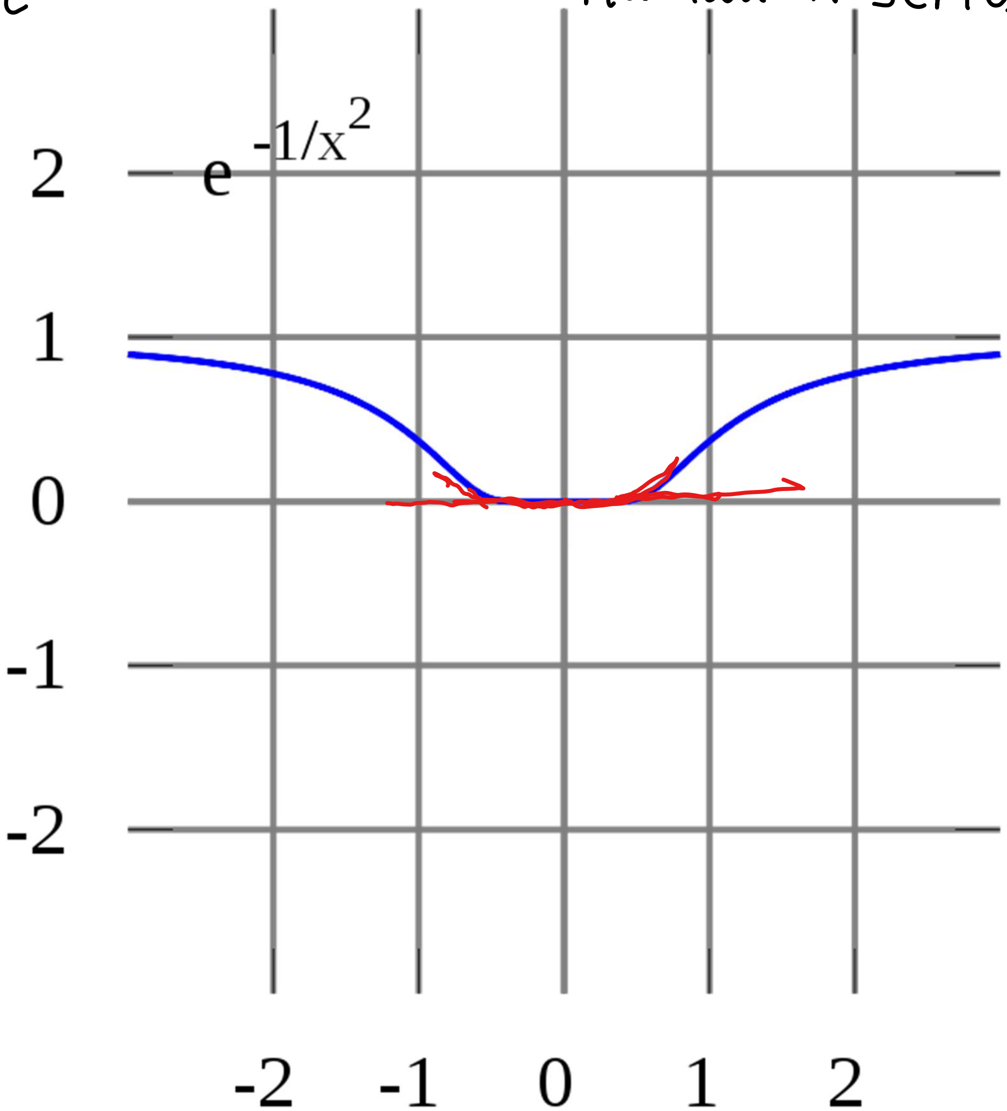
$= 0$

Maclaurin series $= 0$

$f(x) \neq 0$

Maclaurin series

except at $x=0$



#19) $\dfrac{1}{\sqrt{1-x}}$

Step 1: $\dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$, $k = -1/2$

$$\boxed{\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{(-1/2)(-1/2-1)}{2!}x^2 + \dots}$$

$$\frac{1}{\sqrt{1-x}} = 1 - \frac{1}{2}(-x) + \frac{3/4}{2!}(-x)^2 + \dots$$

35

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots + (-1)^n \frac{u^{2n}}{(2n)!} + \dots$$

$u = x^{3/2}$

$u^2 = x^3$, $u^6 = x^9$