Lecture: 04/23 Calculus lecture.

Quiz # 24 (Solutions)

a) $f(x) = \dfrac{3}{3x+4} = \dfrac{3/4}{1+\frac{3}{4}x} = \dfrac{a}{1-r}$, so

$f(x) = \dfrac{3}{4}\sum_{n=0}^{\infty}\left(-\dfrac{3}{4}x\right)^n = \sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3}{4}\right)^{n+1}x^n$, when $\left|\frac{3}{4}x\right|<1$

Interval of convergence: $|x| < 4/3$ or $(-4/3, 4/3)$

b) $g(x) = \dfrac{1}{x+1} = \dfrac{1}{-1+x+2} = -\dfrac{1}{1-(x+2)} = \dfrac{a}{1-r}$

$g(x) = -\sum_{n=0}^{\infty}(x+2)^n = -\sum_{n=0}^{\infty}(x+2)^n$, when $|x+2|<1$

or $|x+2|<1$. Interval of convergence: $(-3,-1)$.

Homework questions.

28. $g(x) = e^{\boxed{-3x}}$

$e^{\boxed{u}} = 1 + u + \dfrac{u^2}{2!} + \dfrac{u^3}{3!} + \dots + \dfrac{u^n}{n!} + \dots$

Plug in $u = -3x$

$e^{-3x} = 1 - 3x + \dfrac{(-3x)^2}{2!} + \dots + \dfrac{(-3x)^n}{n!} + \dots$

$= \sum_{n=0}^{\infty}\dfrac{(-3x)^n}{n!} = \sum_{n=0}^{\infty}\dfrac{(-1)^n\cdot 3^n}{n!}x^n$

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \underbrace{\frac{f^{(n)}(0)}{n!}x^n}_{= \frac{(-3)^n}{n!}x^n} + \dots$$

$f(x) = e^{-3x}$ $\qquad$ $f(0) = 1$

$f'(x) = -3e^{-3x}$ $\qquad$ $f'(0) = -3$

$f''(x) = (-3)^2 e^{-3x}$

$\vdots$

$f^{(n)}(x) = (-3)^n e^{-3x}$, $\quad f^{(n)}(0) = (-3)^n$

#4 $f(x) = \sin x$, $\quad c = \pi/4$

$$f(\pi/4) + \frac{f'(\pi/4)}{1!}(x - \pi/4) + \frac{f''(\pi/4)}{2!}(x - \pi/4)^2 + \dots$$

$$+ \frac{f^{(n)}(\pi/4)(x - \pi/4)^n}{n!} + \dots = \frac{\sqrt{2}}{2}\Big(1 + 1\cdot(x - \pi/4)$$

$$- \frac{1}{2!}(x - \pi/4)^2$$

$$- 1/3!\,(x - \pi/4)^3$$

$$+ \frac{1}{4!}(x - \pi/4)^4 +$$

$$\dots\dots\Big)$$

$f = \sin x$ $\qquad$ $f(\pi/4) = \sin \pi/4 = \frac{\sqrt{2}}{2}$

$f' = \cos x$ $\qquad$ $f'(\pi/4) = \frac{\sqrt{2}}{2}$

$f'' = -\sin x$ $\qquad$ $f''(\pi/4) = -\frac{\sqrt{2}}{2}$

$f''' = -\cos x$ $\qquad$ $f'''(\pi/4) = -\sqrt{2}/2$

$\vdots$

$$= \sum_{n=0}^{\infty}$$

Sec. 9.10. Taylor and Maclaurin series (continued)

Suppose $f(x)$ is a function that can be differentiated an arbitrary number of times at $x=c$. Then it's

- Taylor series of $f(x)$ centered at $x=c$ is

$$f(c)+\frac{f'(c)}{1!}(x-c)+\frac{f''(c)}{2!}(x-c)^2+\dots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\dots$$

coefficient

$$=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n.$$

- Maclaurin series of $f(x)$ is

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\dots+\frac{f^{(n)}(0)}{n!}x^n+\dots$$

$$=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

Examples.

Function f		Maclaurin series	Interval of convergence
e^x	$=$	$\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+\dots$	$(-\infty,+\infty)$
$\sin x$	$=$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$	$(-\infty,+\infty)$
$\cos x$	$=$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots$	$(-\infty,+\infty)$
$\ln(1+x)$	$=$	$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$	$(-1,1]$
$\arctan x$	$=$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\dots$	$[-1,1]$

- In all this cases functions are equal their Maclaurin series on its interval of convergence.
- If $f(x)$ has a power series expansion centered at $x = c$, then this power series expansion is it's Taylor series.

Ex: From Quiz # 24

a) $$\frac{3}{3x+4} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{4}\right)^{n+1} x^n$$ on $(-4/3, 4/3)$.

Maclaurin series of $f(x) = \frac{3}{3x+4}$

b) $$\frac{1}{1+x} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{3^{n+1}}}$$ on $(-1, 5)$

at $x = 2$

Taylor series of $g(x) = \frac{1}{1+x}$ centered at $x = 2$.

Question: $g(x) = \frac{1}{1+x}$, what is $g^{(7)}(2)$ = 7th derivative of g at 2.

Methods of finding Taylor / Maclaurin series:

1. Use the formula $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$
2. Use power series expansion and may be integration.
3. Use standard series at tricks.

$$\frac{1}{1+x} = f(2) + \frac{f'(2)}{1!}(x-2) + \dots + \frac{f^{(7)}(2)}{7!}(x-2)^7 + \dots$$

$$f^{(7)}(2) = \frac{(-1)\,7!}{3^8}$$

$$\frac{(-1)^7 \;(x-2)^7}{3^{7+1}}$$

Examples: a) Find Maclaurin series for $f(x) = x^2 \sin(x^3)$

Standard series

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots + (-1)^n \frac{u^{2n+1}}{(2n+1)!} \cdots$$

true for all u

$$x^2 \sin x^3 = x^2 \left(x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} - \cdots + (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} \cdots \right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1+2}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$$

b) $\cos^2 x = \dfrac{\cos(2x)+1}{2}$

Maclaurin series.

$$\cos u = 1 - \frac{u^2}{2!} + u^4/4! - \cdots + \frac{(-1)^n u^{2n}}{(2n)!} + \cdots$$

$-\infty < u < \infty$.

$$\cos^2 x = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n \cdot x^{2n}}{(2n)!} \quad -\infty < x < +\infty.$$

c) e^x centered at $x = 3$

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \quad -\infty < u < \infty.$$

$$e^{x-3+3} = e^3 \cdot e^{(x-3)} \quad \text{"} u$$

$$= e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!} \quad -\infty < x < \infty,$$

d) $f(x) = x^2 - x + 3$ Taylor series centered at $x = 1$.

$f(1) = 1 - 1 + 3 = 3$

$f'(x) = 2x - 1$ $\qquad f'(1) = 1$

$f''(x) = 2$ $\qquad f''(1) = 2$

$f'''(x) = 0$ $\qquad f^{(3)}(1) = 0$

$\vdots$ $\qquad \vdots$

$$x^2 - x + 3 = 3 + 1(x-1) + \frac{2}{2!}(x-1)^2$$
$$= 3 + x - 1 + (x^2 - 2x + 1)$$
$$= x^2 - x + 3.$$

If $f(x)$ is a polynomial, it is its own Taylor series (and power series)

<u>Ex:</u> Maclaurin series of

$$f(x) = \sqrt{1+x} = (1+x)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}$$
$$f''(x) = \frac{1}{2}\left(-\tfrac{1}{2}\right)(1+x)^{-3/2}$$
$$f^{(3)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-5/2}$$
$$\vdots$$
$$\boxed{f^{(n)}(x) = (-1)^{n+1}\,\frac{1\cdot 1\cdot 3\cdot 5\cdot 7\ldots(?)}{2^n}\cdot(1+x)^{-?}}$$