Lecture: 04/22 Proof lecture

Hwk 21 # 3 a) Since A and B are countable, there are bijections $f: \mathbb{N} \to A$ and $g: \mathbb{N} \to B$. Let $a_n = f(n)$ and $b_n = g(n)$ $\forall n \in \mathbb{N}$. Then $A = \{a_1, a_2, \dots, a_n, \dots\}$ and $B = \{b_1, b_2, \dots, b_n, \dots\}$.

The set $C = A \cup B$ can be arranged as a list $C = \{a_1, b_1, a_2, b_2, \dots, a_n, b_n, \dots\}$. Since $A \cap B = \emptyset$, $a_i \neq b_j$ for all $i, j \in \mathbb{N}$.

Define a bijection $h: \mathbb{N} \to C$ by

$$h(m) = \begin{cases} f(k) = a_k, & \text{if } m = 2k-1,\ k \in \mathbb{N}; \\ g(k) = b_k, & \text{if } m = 2k,\ k \in \mathbb{N}. \end{cases}$$

Thus $C = A \cup B$ is countable.

3 b) Proof by induction:

Base step: $n = 1$: A_1 is countable

$n = 2$: A_1, A_2 are countable, $A_1 \cap A_2 = \emptyset \Rightarrow$ $A_1 \cup A_2$ is countable (see 3 a)

Induction step: Assume that: if $A_1, A_2, \dots, A_k$ are countable and $A_i \cap A_j = \emptyset$, then $A_1 \cup A_2 \cup \dots \cup A_k$ is countable.

Let $A_1, A_2, \dots, A_k, A_{k+1}$ be countable and $A_i \cap A_j = \emptyset$, and let $A = A_1 \cup A_2 \cup \dots \cup A_k$ and $B = A_{k+1}$.

A and B are countable by assumption, $A \cap B =$

$= (A_1 \cup A_2 \cup \dots \cup A_k) \cap B = (A_1 \cap A_{k+1}) \cup \dots \cup (A_k \cap A_{k+1}) = \emptyset$

Then by 3 a) $A \cup B = A_1 \cup \ldots \cup A_k \cup A_{k+1}$ is countable.

Homework questions?

Chapter 21. Infinity.

Recall:

Definition: A set A is called countable if there is a bijection $f: \mathbb{N} \to A$

Examples:
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable.
- If A is countable and B is finite, then $A \cup B$ is countable.
- If $A_1, A_2, \ldots, A_n$ are countable, then $A_1 \cup A_2 \cup \ldots \cup A_n$ is countable.
- $A_1, A_2, \ldots, A_n, \ldots$ are countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Finite or countable unions of countable sets are countable

Proposition 21.2 Every infinite subset of $\mathbb{N}$ is countable.

Proof: Let S be an infinite subset of $\mathbb{N}$.
Let s_1 be the smallest integer in $\mathbb{N}$
Let s_2 be the smallest integer in $S - \{s_1\}$
...
Let s_n be the smallest integer in $S - \{s_1, \ldots, s_{n-1}\}$
...

Then $S = \{s_1, \ldots, s_n, \ldots\}$ and the bijection is $f: \mathbb{N} \to S$ $\quad f(n) = s_n$.

Theorem (Cantor): $(0,1)$ is not countable.

Note: Since there is a bijection from $(0,1)$ to $\mathbb{R}$, $\mathbb{R}$ is also not countable.

Proof by contradiction

Proof: Suppose $(0,1)$ is countable. Then one can arrange all real numbers on $(0,1)$ as an infinite list

$$(0,1) = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

We will find a number $b \in (0,1)$ which is not on the list. This will be a contradiction

Every real number $x \in (0,1)$ can be written in a unique way as a decimal

$$a_1 = .a_{11} a_{12} a_{13} \ldots a_{1k} \ldots$$
$$a_2 = .a_{21} a_{22} a_{23} \ldots a_{2k} \ldots$$
$$a_3 = .a_{31} a_{32} a_{33} \ldots a_{3k} \ldots$$
$$\vdots$$
$$a_n = .a_{n1} a_{n2} a_{n3} \ldots a_{nk} \ldots a_{nn}$$
$$\vdots$$

(Ex: $a_5 = .123123\ldots$ $\quad a_{54} = 1$

$= .a_{51}a_{52}a_{53}a_{54}a_{55}\ldots$

We will write $.1\underbrace{999\ldots}$ $= .200000$)

Now we will constract a number that is not on the list: $b \in (0,1)$

$$b = .b_1 b_2 \ldots b_n \ldots$$

$$b_1 = \begin{cases} 3 & \text{if } a_{11} \neq 3 \\ 4 & \text{if } a_{11} = 3. \end{cases}$$

$$b_2 = \begin{cases} 3 & \text{if } a_{22} \neq 3 \\ 4 & \text{if } a_{22} = 3 \end{cases}$$

$\vdots$

$$b_n = \begin{cases} 3 & \text{if } a_{nn} \neq 3 \\ 4 & \text{if } a_{nn} = 3 \end{cases}$$

$\vdots$

Claim: $b \neq a_1$ since $a_{11} \neq b_1$

$b = .b_1 b_2 \ldots$ $\quad a_1 = .a_{11}a_{12}\ldots$

$b \neq a_2$ since $b_2 \neq a_{22}$

$a_2 = .a_{21}\underbrace{a_{22}}\ldots$

$b = .b_1\underbrace{b_2}\ldots$

$b \neq a_3 \qquad b_3 \neq a_{33}$

. . . .

$b \neq a_n$ since $b_n \neq a_{nn}$

(n^{th} digit of b is different than n^{th} digit of a_n)

. . . . . .

So b is <u>not</u> on the list $\{a_1, a_2, a_3, \dots, a_n \dots\}$

<u>contradiction</u>.

Cantor diagonalization argument.

<u>A Hierarhy of Infinities.</u>

<u>Def</u>: Let A and B be two sets, then we say that

- $\mathrm{card}(A) = \mathrm{card}(B)$ if there is a bijection $f: A \to B$.
- $\mathrm{card}(A) \geq \mathrm{card}(B)$ if there is a surjection $g: A \to B$.

- $\mathrm{card}(A) \leq \mathrm{card}(B)$ if there is a injection $h: A \to B$.

We have an injective $h: \mathbb{N} \to \mathbb{R}$, namely $h(n) = n$. So $\mathrm{card}(\mathbb{N}) \leq \mathrm{card}(\mathbb{R})$

We also know that $\mathbb{R}$ is not countable, so $\mathrm{card}(\mathbb{N}) < \mathrm{card}(\mathbb{R})$ $(\mathrm{card}(\mathbb{N}) \neq \mathrm{card}(\mathbb{R}))$

$\mathrm{card}(\mathbb{R}) = \aleph_1 \leftarrow$ cardinality of continuum

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<u>Q</u>: Is there a set A such that $\mathrm{card}(\mathbb{N}) < \mathrm{card}(A) < \mathrm{card}(\mathbb{R})$?

Coen. in 1950's. Forsing.