Lecture: 04/21 Calculus lecture.

Homework questions.

14 p. 662.

$$g(x) = \frac{3x-8}{3x^2+5x-2} \quad \text{at } c=0$$

$$g(x) = \frac{3x-8}{(3x-1)(x+2)} = \frac{A}{3x-1} + \frac{B}{x+2}$$

$$A(x+2) + B(3x-1) = 3x-8$$

$$x=-2; \quad B\cdot(-7) = -14, \quad \boxed{B=2}$$

$$x = 1/3; \quad A\left(\tfrac{1}{3}+2\right) = 1-8 = -7$$

$$7/3\, A = -7, \quad \boxed{A=-3}$$

$$g(x) = \frac{-3}{3x-1} + \frac{2}{x+2} = \frac{3}{1-3x} + \frac{1}{1+\frac{x}{2}}$$

$$= 3\sum_{n=0}^{\infty}(-3x)^n + \sum_{n=0}^{\infty}(-x/2)^n$$

$$\frac{a}{1-r}$$

$$|3x| < 1 \qquad |x/2| < 1$$

Interval of conv. $\rightarrow$ $\boxed{-1/3 < x < 1/3}$ $\qquad -2 < x < 2$

Quiz solutions.

1a) Step 1. Ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+2)^{n+1}}{(n+1)\cdot 4^{n+1}}\Big/\left(\frac{(-1)^n\,(x+2)^n}{n\cdot 4^n}\right)\right| = \lim_{n\to\infty}\left|\frac{(x+2)}{4}\,\frac{n}{n+1}\right|$$

$$= \left|\frac{(x+2)}{4}\right| < 1 \Rightarrow |x+2| < 4$$

$\boxed{\text{Radius of convergence } R = 4.}$

Series converges absolutely when $-4 < x+2 < 4$, $-6 < x < 2$

Step 2. Checking endpoints: $x = -6 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{n\,4^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

$x = 2 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n\cdot 4^n}{n\cdot 4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \leftarrow$ alt. series (converges)

$\boxed{\text{Interval of convergence: } (-6, 2]}$

b) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n\cdot \cancel{n}(x+2)^{n-1}}{\cancel{n}\cdot 4^n}$. Checking endpoints:

$x = -6 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n\cdot(-4)^{n-1}}{4^n} = \frac{1}{4}\sum_{n=1}^{\infty}(-1) \leftarrow$ (diverges, by divergence test)

$x = 2 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n\;4^{n-1}}{4^n} = \frac{1}{4}\sum_{n=1}^{\infty}(-1)^n$

$\boxed{\text{Interval of convergence: } (-6, -2)}$

c) $\int f(x)\,dx = \int \sum_{n=1}^{\infty}\frac{(-1)^n(x+2)^n}{n\;4^n}\,dx = \sum_{n=1}^{\infty}\frac{(-1)^n(x+2)^{n+1}}{n(n+1)\cdot 4^n}$

$x = -6 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n(-4)^{n+1}}{n(n+1)\cdot 4^n} = -4\sum_{n=1}^{\infty}\frac{1}{n(n+1)} \leftarrow$ conv direct comparison test. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} < \sum_{n=1}^{\infty}\frac{1}{n^2}$.

$x = -2 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)} \leftarrow$ conv by absolute convergence test.

$\boxed{\text{Interval of convergence: } [-6, -2]}$.

Sec. 9.10. Taylor and Maclaurin Series.

Definition: Suppose f is represented by a power series:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n,$$

which is convergent on some interval containing c. Then the series is called the Taylor series of $f(x)$ centered at $x=c$.

If $c=0$, i.e. if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \leftarrow$$ Taylor series of $f(x)$ centered at $x=0$ or Maclaurin series of $f(x)$.

Assume

$$(*) \quad \boxed{f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots}$$

Our goal is to find coefficients of a Taylor series expansion: $a_0, a_1, a_2, \dots, a_n = ?$

a_0? Plug in $x=c$: $f(c) = a_0 + a_1(\underbrace{\cancel{c-c}}_{0}) + \dots$

$$\boxed{a_0 = f(c)}$$

a_1? Take derivative of both sides of $(*)$

$$f'(x) = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$$

$$f'(c) = a_1 \qquad \boxed{a_1 = f'(c)}/1!$$

a_2? $f''(x) = 2a_2 + 3\cdot 2\, a_3(x-c) + \dots$

$$f''(c) = 2a_2 \qquad \boxed{a_2 = \frac{f''(c)}{2\cdot 1}} = 2!$$

a_3? $$a_3 = \frac{f'''(c)}{3!}$$

$$a_4 = \frac{f^{(4)}(c)}{4\cdot 3\cdot 2\cdot 1} = \frac{f^{(4)}(c)}{4!}$$

$$f^{(n)}(c) = n!\, a_n, \qquad \boxed{a_n = \frac{f^{(n)}(c)}{n!}}$$

Taylor series of $f(x)$ centered at $x=c$ is

$$f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

Maclauren series: $c=0$:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

Note: All infinitely differentiable functions have Taylor series, but not all functions are equal to their Taylor series.

Functions which are equal to their Taylor series are called real analytic (most of functions we use).

Computing Maclauren series of some useful functions (see p. 670)

$$f(x) = e^x = \sum_{n=0}^{\infty} a_n x^n$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

$$f(x) = e^x, \quad f^{(n)}(x) = e^x, \quad f^{(n)}(0) = e^0 = 1$$

$$e^x = 1/\underbrace{0!}_{=1} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots + \frac{1}{n!}x^n + \dots$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

This series converges on $(-\infty, +\infty)$

Ex: $f(x) = \sin(x)$

$\boxed{f(0) = \sin 0 = 0}$

$f'(x) = \cos x$, $\boxed{f'(0) = 1}$

$\boxed{f''(0) = 0}$, $f''(x) = -\sin x$

$\boxed{f^{(3)}(0) = -1}$, $\boxed{f^{(4)}(0) = 0}$ ...

$$\sin x = 0 + \underbrace{\frac{1}{1!}x}_{} + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} = (-1)^{1-1}\frac{x^{2\cdot 1-1}}{(2\cdot 1-1)!} + (-1)^{2-1}\frac{x^{2\cdot 2-1}}{(2\cdot 2-1)!} \dots = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\boxed{\sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}} \quad -\infty < x < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$\cos x = f(x)$

$f(0) = 1 \qquad f''(0) = -1 \qquad f^{(4)}(0) = 1 \ldots$

$f'(0) = 0 \qquad f^{(3)}(0) = 0$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad -\infty < x < +\infty .$$

<u>Ex:</u> Use the definition to compute Taylor series centered at $c = 2$.

$f(x) = \ln x$

$$\ln x = \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$a_n = \frac{f^{(n)}(c)}{n!} = \frac{f^{(n)}(2)}{n!}$$

$f(x) = \ln x$

$f'(x) = \frac{1}{x} = x^{-1}$

$$f''(x) = -\frac{1}{x^2} = -x^{-2}$$

$$f'''(x) = (-1)(-2)x^{-3}$$

$$f^{(n)}(x) = \underbrace{(-1)(-2)\ldots(-(n-1))}_{n-1}x^{-n}$$

$$f^{(n)}(2) = (-1)^{n-1}\cdot(n-1)!\cdot\frac{1}{2^n}, \quad a_n = \frac{(-1)^{n-1}(n-1)!}{2^n\cdot n!}$$

$$\ln x = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{(n-1)!}{n!}\frac{1}{2^n}(x-2)^n$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(x-2)^n}{n\cdot 2^n}}$$

converges on $(0, 4]$

$$(\arctan 2x)' = \frac{2}{1+(2x)^2} = \frac{a}{1-r}$$

$$a = 2, \quad r = -4x^2$$

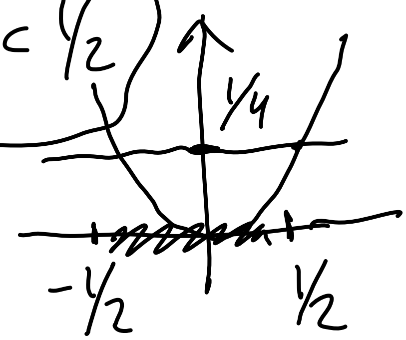
$$(\arctan 2x)' = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 2(-4x^2)^n$$

$$= 2\sum_{n=0}^{\infty} (-1)^n \cdot 4^n x^{2n}$$

$$|r| = |4x^2| = 4x^2 < 1$$

$$\boxed{x^2 < 1/4} \Rightarrow$$

$$\boxed{-1/2 < x < 1/2}$$



$$\arctan 2x =$$

$$= 2\int \sum_{n=0}^{\infty} (-1)^n \cdot 4^n \cdot x^{2n}\, dx$$

$$= \boxed{2\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n \cdot x^{2n+1}}{2n+1}} + C$$

$x = 0$: $\arctan 0 = C$, $C = 0$

Test endpoints: $x = -1/2$ ← converges

$x = 1/2$ ← converges.

$\boxed{[-1/2, 1/2]}$ ← int. of convergence.