Lecture: Sec. 21. Infinity (continued)

More example of equivalence relations:

<u>Ex</u>: Set $S = \mathbb{R}$. Equivalence relation is $x \sim y$ if $x - y =$ integer.

$$\overline{x - z = x - y + y - z} = m + n \leftarrow \text{integer}.$$

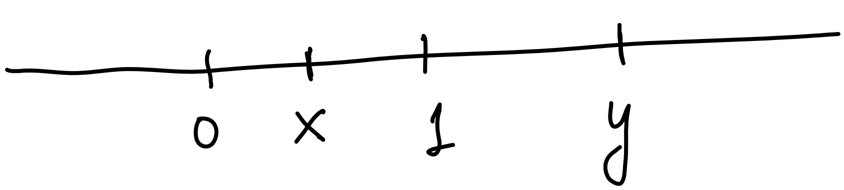
What are equivalence classes:

$cl(0) = \mathbb{Z}$

$cl(1/2) = \{n + 0.5 \mid n \in \mathbb{Z}\}$.

Take any number $0 \leq r < 1$

$cl(r) = \{r + n \mid n \in \mathbb{Z}\}$.



Given any $y \in \mathbb{R}$, there is always $x \in [0,1)$

s.t. $y \in cl(x)$.

Let. $y = 3.715$, $y \in cl(.715)$

$y = -.333\ldots = -.\overline{3} = -\frac{1}{3}$, $y \in cl(.666\ldots = .\overline{6} = \frac{2}{3})$

Equivalence classes are in one-to-one and onto corresp. with points on $[0,1)$

Main example

Let S be the set of all possible sets Let $A, B \in S$ (i.e. A and B are two sets), then say that $A \sim B$ (A and B are equivalent) if $|A| = |B|$ (or $card(A) = card\, B$).

Proposition 21.1 The relation described above is an equivalence relation

Proof: ① $A \sim A$ (reflexivity).

(Recall that $|A| = |B|$ means that there is a bijection $f: A \to B$)

$A \sim A \iff$ there is a bijection $f: A \to A$, namely $f(a) = a \ \forall a \in A$.

② Symmetricity: $A \sim B \Rightarrow B \sim A$, i.e. If there is a bijection $f: A \to B$, then there is a bijection $g: B \to A$.

If $f: A \to B$ is a bijection, then define $g = f^{-1}$ (inverse function exists for bijections)

Need to know that g is also a bijection (homework problem).

③ Transitivity: $A \sim B$ and $B \sim C \Rightarrow A \sim C$. I.e. if $f: A \to B$ is a bijection and if $g: B \to C$ is a bijection, then there is a

bijection $h = g \circ f : A \to C$
(Proved that composition of two bijections is a bijection last time).

Equivalence classes:

$cl(\emptyset) = \{\emptyset\}$.

$cl(\underbrace{\{1\}}_{\text{set with one element } 1}) = cl(\underbrace{\{\sqrt{2}\}}_{\text{set with one element } \sqrt{2}}) \ldots$

⋮

$cl\{\{1, 2, \ldots, n\}\} =$ all sets with n elements.

⋮

$cl(\mathbb{N}) =$ all countable sets

$\mathbb{N} \in \underbrace{cl(\mathbb{N})}_{\text{all sets of the same size as } \mathbb{N}}, \quad \mathbb{Z} \in cl(\mathbb{N})$

Countable sets.

Definition: A set B is called countable, if there is a bijection $f: \mathbb{N} \to B$. ($\mathbb{N} \sim B$, or $B \in cl(\mathbb{N})$, or $|B| = |\mathbb{N}|$, $card(B) = card(\mathbb{N}) = \aleph_0$).

Remarks: 1) All countable sets are infinite.

2) A set A is countable if and only if all elements of A can be arranged as an infinite list

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

Then the bijection $f: \mathbb{N} \to A$ is $f(n) = a_n$.

What about $\mathbb{Q}$? Is it countable or not?

Start with positive rational numbers.

m/n, $m, n \in \mathbb{N}$

$$\begin{array}{ccccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots & \frac{m}{1} & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \cdots & & \frac{m}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \cdots & & \frac{m}{3} & \cdots \\ \vdots & \vdots & \vdots & & & & \\ \frac{1}{n} & \frac{2}{n} & \frac{3}{n} & \cdots & & \frac{m}{n} & \cdots \\ \vdots & & & & & & \end{array}$$

contains all positive rational numbers.

$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \cancel{\frac{2}{2}}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \cancel{\frac{2}{4}}, \ldots$

$\underbrace{\quad}_{m+n=2}$ $\underbrace{\quad}_{m+n=3}$ $\underbrace{\quad}_{m+n=4}$ $\underbrace{\quad}_{m+n=5}$ $\underbrace{\quad}_{m+n=6}$

$\dots \quad \underbrace{\quad}_{m+n=k} \quad \dots$

Remove all double counted numbers from the list

So the list is a bijection

$f: \mathbb{N} \to \mathbb{Q}_+$

positive rational

$0, \frac{1}{1}, -\frac{1}{1}, \frac{2}{1}, -\frac{2}{1}, \frac{3}{1}, -\frac{3}{1}, \frac{1}{3}, -\frac{1}{3}, \dots$

So this gives a bijection

$f: \mathbb{N} \to \mathbb{Q}$.

So $\boxed{|\mathbb{N}| = |\mathbb{Q}|}$!!!