Lecture: 04/20 Calculus lecture.

Homework questions.

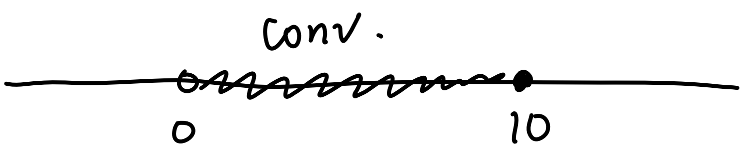
#46 on p. 654

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n\,5^n}$$

a) Ratio test

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-5)^{\cancel{n}+1}}{(n+1)\,5^{n+1}} \Big/ \frac{(-1)^{n+1}\cancel{(x-5)^n}}{n\cdot 5^n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)}{5}\,\frac{n}{n+1}\right|$$

$$= \left|\frac{x-5}{5}\right| < 1, \quad |x-5| < 5, \quad -5 < x-5 < 5$$



$$\boxed{R = 5}$$

Endpoints;

$x = 0$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-5)^n}{n\cdot 5^n} = \sum_{n=1}^{\infty}\frac{\overbrace{(-1)^{n+1}\cdot(-1)^n}^{=-1}}{n}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n} \leftarrow \text{diverges.}$$

$x = 10$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot \cancel{5^n}}{n\,\cancel{5^n}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \swarrow \text{alt. conv.}$$

Interval of convergence is $(0, 10]$

b) $$f'(x) = \left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-5)^n}{n\cdot 5^n}\right)' = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cancel{n}(x-5)^{n-1}}{\cancel{n}\cdot 5^n}$$

Radius of conv. does not change:

$x = 0$: $\displaystyle\sum_{n=1}^{\infty}\underbrace{(-1)^{n+1}(-1)^n}_{-1}$ ↑ diverges.

Int. of conv. $(0, 10)$

$x = 10$ $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}$ ↖ diverges.

Sec. 9.9. Representation of functions by power series.

Example 1. Power series $\sum_{n=0}^{\infty} x^n = 1+x+x^2+\dots+x^n+\dots$ $= a+ar+ar^2+\dots+ar^n+\dots$ $= \begin{cases} \frac{a}{1-r}, & |r|<1 \\ \text{diverges}, & |r|\geq 1. \end{cases}$

is geometric with $r = x$, so

$$\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x}, & \text{if } |x|<1 \\ \text{diverges}, & \text{if } |x|\geq 1 \end{cases}$$

Conclusions: Interval of convergence of this series is $(-1, 1)$ and on this interval

$$\frac{1}{1-x} = 1+x+x^2+\dots+x^n+\dots = \sum_{n=0}^{\infty} x^n.$$

$$\boxed{\frac{a}{1-r} = a+ar+ar^2+\dots+ar^n+\dots \quad \text{when } |r|<1.}$$

Example: Represent the following functions as a geometric power series centered at $c=0$. What is its interval of convergence.

a) $\frac{2}{3+4x} = f(x)$ $\qquad \frac{2}{3+4x} = \frac{a}{1-r}$

$$\frac{2}{3+4x} = \frac{2}{3(1+\frac{4}{3}x)} = \frac{2/3}{\underbrace{1+4/3x}_{1-\underbrace{(-4/3x)}_{r}}}, \quad a = 2/3, \quad r = -\frac{4}{3}x$$

$$\frac{2}{3+4x} = \frac{2}{3} + \frac{2}{3}\cdot\left(-\frac{4}{3}x\right) + \ldots + \frac{2}{3}\left(-\frac{4}{3}x\right)^n + \ldots$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{2}{3}\cdot\left(-\frac{4}{3}\right)^n}_{a_n}\cdot x^n, \quad \text{conv. when } |r| = \left|-\frac{4}{3}x\right| < 1$$

$$\frac{4}{3}|x| < 1, \quad |x| < \frac{3}{4}$$

Radius of convergence is $R = \frac{3}{4}$, interval of conv. is $\left(-\frac{3}{4}, \frac{3}{4}\right)$

On $\left(-\frac{3}{4}, \frac{3}{4}\right)$

$$\frac{2}{3+4x} = \sum_{n=0}^{\infty} \frac{2}{3}\left(-\frac{4}{3}\right)^n x^n$$

$$\boxed{\frac{2}{3+4x} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{3^{n+1}} x^n.}$$

Finding power series of $f(x) = \ln(1+x)$ by integration

b) $$\left(\ln(1+x)\right)' = \frac{1}{1+x} = \frac{a}{1-r}$$

$$\boxed{a = 1, \quad r = -x}$$

$$\frac{1}{1+x} = \left(\ln(1+x)\right)' = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\boxed{-1 < x < 1}$$

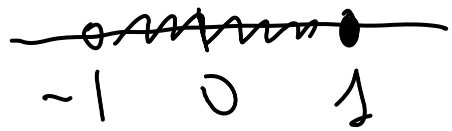
$$\underline{\ln(1+x)} = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx = \int \frac{1}{1+x} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + \cancel{C}^{=0}.$$

$x = 0$: $\quad \underset{\overset{\shortparallel}{0}}{\ln(1)} = C$

Thus, $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ for $-1 < x < 1$.

We need to check endpoints:



$x = 1$: $\quad \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n+1} \leftarrow$ alt. converges

$x = -1$: $\quad \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{n+1}}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges

$(-1)^n(-1)^{n+1} = (-1)^{2n+1} = (-1)^{odd} = -1$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{m+1} \frac{x^m}{m} \ldots$$

$\boxed{n+1 = m}$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m} \text{ on } \underline{(-1, 1]}$$

In particular, when $x = 1$

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{m+1} \frac{1}{m} \ldots$$

Keep in mind:

$$\frac{a}{1-r} = \sum_{r=0}^{\infty} a r^n \text{ when } |r| < 1 \ (-1 < r < 1).$$

Example: Find the power series representation and interval of convergence.

a) Of function $f(x) = \frac{1}{1+2x}$ centered at $x = 3$

$$\boxed{\frac{1}{1+2x} = \sum_{n=0}^{\infty} a_n (x-3)^n}$$ ← what we want.

Algebra:

$$\frac{1}{1+2x} = \frac{1}{1+2(x-3+3)} = \frac{1}{1+2(x-3)+6} = \frac{1}{7+2(x-3)} =$$

$$\frac{1}{1+2x} = \frac{a}{1-r}$$

must contain $(x-3)$

Interval of convergence. $|r|<1$, $\left|\frac{2}{7}(x-3)\right|<1$

$|x-3|<7/2$

$$\frac{1}{1+2x} = \frac{1}{7\left(1+\frac{2}{7}(x-3)\right)} = \frac{1/7}{1+\frac{2}{7}(x-3)} = \frac{a}{1-r}$$

$a = \frac{1}{7}$, $\boxed{r = -\frac{2}{7}(x-3)}$

$$\frac{1}{1+2x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{7}\left(-\frac{2}{7}(x-3)\right)^n = \boxed{\frac{1}{7}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{7}\right)^n (x-3)^n}$$

$-7/2 < x-3 < 7/2$

$-\frac{1}{2} < x < 13/2$

b) Of function $g(x) = \frac{5}{4-2x}$ centered at $x = -5$:

$$g(x) = \frac{5}{4-2x} = \frac{5}{4-2(x+5-5)} = \frac{5}{14-2(x+5)} = \frac{5}{14\left(1-\frac{2}{7}(x+5)\right)}$$

$$g(x) = \frac{5/14}{1-\frac{2}{7}(x+5)} = \frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$$

$a = 5/14$, $r = \frac{2}{7}(x+5)$

$$g(x) = \sum_{n=0}^{\infty} \frac{5}{14} \cdot \left(\frac{2}{7}(x+5)\right)^n, \text{ when } \left|\frac{2}{7}(x+5)\right| < 1.$$

$$\boxed{g(x) = \frac{5}{14} \sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n (x+5)^n} \text{ when } |x+5| < 7/2 \text{ or}$$

$$-7/2 < x+5 < 7/2$$

Interval of convergence. $\rightarrow$ $\boxed{-\frac{17}{2} < x < -3/2}$

Operations with power series.

One can add, subtract, multiply, and even divide converging power series.

Example: Find the power series centered at $x=0$ of $f(x) = \frac{2x+4}{x^2+4x+3}$. What is its interval of convergence?

Solution: Partial fractions.

$$\underset{\sum_{n=0}^{\infty} a_n x^n}{\overset{\parallel}{f(x)}} = \underbrace{\frac{2x+4}{(x+3)(x+1)}} = \underbrace{\frac{1}{x+3} + \frac{1}{x+1}}$$

$$\frac{1}{x+3} = \frac{a}{1-r} \; ; \; \frac{1}{3(1+x/3)} = \frac{1/3}{1+x/3}$$

$$a = 1/3, \; r = -x/3$$

$$\frac{1}{x+3} = \sum_{n=0}^{\infty} a r^n = \sum_{n=0}^{\infty} 1/3 \left(-x/3\right)^n, \; |r| = |-x/3| < 1.$$

$$\boxed{\frac{1}{x+3} = \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{3^n}, \quad |x| < 3 \text{ or on } (-3,3)}$$

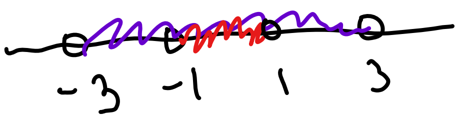
$$\frac{1}{x+1} = \frac{a}{1-r} = \sum_{n=0}^{\infty} 1\cdot(-x)^n, \quad |r| = |-x| < 1 \text{ on } (-1,1)$$

$a = 1$, $r = -x$

$$f(x) = \frac{1}{x+3} + \frac{1}{x+1} = \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n} + \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{3^{n+1}} + 1\right) x^n \quad \text{on } (-1,1)$$

↑ smaller of the two.



#28. Ratio test: $\sum \frac{(-1)^n x^{2n}}{n!}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(n+1)!} \Big/ \frac{(-1)^n x^{2n}}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2\, n!}{(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{x^2}{n+1} \right| = 0 < 1$$

Always converges. Int. of conv. $(-\infty, +\infty)$.

46 d) $\int f(x)\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n\,5^n}\,dx$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,(x-5)^{n+1}}{n \cdot 5^n\,(n+1)}$$

$x = 0$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-5)^{n+1}}{n(n+1)\cdot 5^n} =$

$$= \quad 5 \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$ ← converges by dir. comparison with $5 \sum \frac{1}{n^2}$

$x = 10$ Same.

Interval of convergence is $[0, 10]$

$f(x) \mapsto (0, 10]$ $\qquad \int f(x)\,dx \mapsto [0, 10]$

$f'(x) \mapsto (0, 10)$

$f''(x) \mapsto (0, 10)$