**Lecture:** 04/17 Proof lecture

Hwk 19 (comments)

2 c) bijection from $(a,b) \to \mathbb{R}$.

$f : (-\pi/2, \pi/2) \to \mathbb{R}$ a)

$g : (a, b) \to (-\pi/2, \pi/2)$ b)

Then $h = g \circ f : (a,b) \to \mathbb{R}$.

$$h(x) = \left( \frac{1}{x-a} - \frac{1}{x-b} \right)$$

$$\left( \frac{1}{x-a} + \frac{1}{x-b} \right)$$

Why are these onto and one-to-one

$$\frac{1}{x-a} + \frac{1}{x-b} = C \iff \text{quadratic equation.}$$

# Homework questions

## Chapter 21. Infinity.

**Definition.** Sets $A$ and $B$ have the same size (or same cardinality) if there is a bijection $f: A \to B$ ($|A| = |B|$ or $\mathrm{card}(A) = \mathrm{card}(B)$)

**Examples:**

a) $A = \{2n \mid n \in \mathbb{N}\}$, $B = \mathbb{N}$

$f: A \to B$, $\boxed{f(k) = \frac{k}{2}}$

$\mathrm{card}(A) = \mathrm{card}(\mathbb{N}) = \aleph_0$

$$\begin{array}{ccccccc} 2, & 4, & 6, & 8, & \ldots, & 2n, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 1, & 2, & 3, & 4, & \ldots, & n & \end{array}$$

$A \subset B$

b) Bijection between $\mathbb{N}$ and $\mathbb{Z}$?

$$\begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ & \uparrow & \uparrow & \updownarrow & \updownarrow & \uparrow & \\ & 5 & 3 & 1 & 2 & 4 & \end{array} \quad \uparrow h$$

$$h(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ -\dfrac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

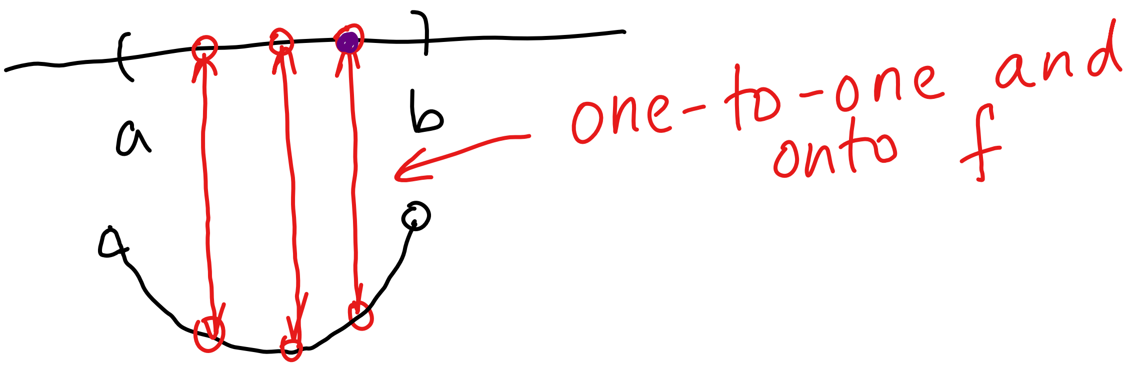
$\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \aleph_0$ aleph zero

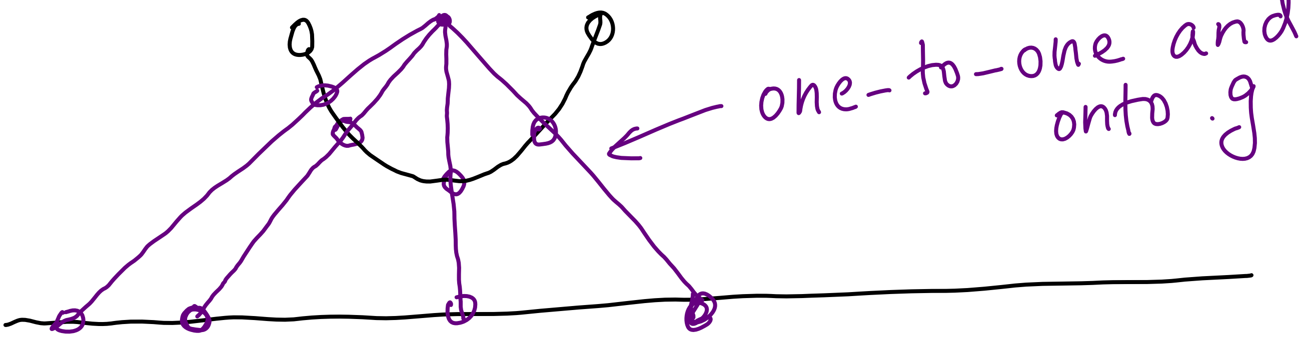
c) At home you showed that

$\text{card}((-\pi/2, \pi/2)) = \text{card}(\mathbb{R})$

$f: (-\pi/2, \pi/2) \to \mathbb{R}$, $f(x) = \tan x$

d) $\text{card}((a,b)) = \text{card}(\mathbb{R})$ (problem 2c in hwk 19)






$g \circ f : (a,b) \to \mathbb{R}$ which is one-to-one and onto

Questions for the future:

card($\mathbb{N}$), card($\mathbb{Q}$), card($\mathbb{R}$)

---

Def (equivalence relation, Chapter 18).

A relation $\sim$ on a collection of objects $S$ is called an equivalence relation if it obeys the following properties

1) $a \sim a$ for all $a \in S$ (reflexivity)
2) $a \sim b \Rightarrow b \sim a$ $\forall a,b \in S$ (symmetric)
3) $a \sim b$ and $b \sim c \Rightarrow a \sim c$ $\forall a,b,c \in S$ (transitivity)

Not equivalence relations:

- A ♡ B   A likes B ← not an equivalence relation.

  $A \sim B$ if

- On $\mathbb{R}$, $a \sim b$ if $a \leq b$.

  $a \sim a$? $a \leq a$ ✓ (reflexive)

  $a \leq b \Rightarrow b \leq a$ No (not symmetric)

  $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitive)

  Not an equivalence relation, not symmetric.

<u>Example</u>: a) Equivalence relation on $S = \mathbb{Z}$.

$m \sim n$ if $m - n$ is divisible by 2

$2 \sim 4$? $2 - 4 = -2$ is div. by 2 Yes

$5 \sim 3$? $5 - 3 = 2$ is div. by 2

$10 \sim 3$? No $10 - 3$ is not div. by 2.

Checking properties:

1) $n \sim n$ ? Yes $n - n = 0$ is div. by 2

2) $m \sim n \Rightarrow n \sim m$ ?

$m \sim n \Leftrightarrow m - n = 2k$, $k \in \mathbb{Z}$

$n \sim m \Leftrightarrow n - m = -2k = 2 \cdot \underbrace{(-k)}_{\in \mathbb{Z}}$.

3) $m \sim n$, $n \sim l \Rightarrow m \sim l$

$\Updownarrow \qquad \Updownarrow$

$m - n = 2k$, $l - n = 2r$,

$m - l = m - n + n - l = 2k + 2r$
$= 2(k + r)$

$\Rightarrow m \sim l$.

---

**Def: Equivalence classes.**

Let $S$ be a set and $\sim$ be an equivalence relation

Then for each $a \in S$, the equivalence class of $a$, $cl(a) = \{b \in S \mid a \sim b\}$.

Back to our example:

$$\mathcal{Cl}(0) = \{0, 2, -2, 4, -4, \dots\}$$

$n \sim 0$

$$\mathcal{Cl}(1) = \{1, -1, 3, -3, \dots\}$$

$\mathbb{Z}$ = disjoint union of $\mathcal{Cl}(0)$ and $\mathcal{Cl}(1)$

$$\mathbb{Z} = \underbrace{\mathcal{Cl}(0)}_{\text{even integers}} \sqcup \underbrace{\mathcal{Cl}(1)}_{\text{odd integers}}$$

$$\mathcal{Cl}(4) = \mathcal{Cl}(0)$$
$$\mathcal{Cl}(5) = \mathcal{Cl}(1)$$

<u>Ex:</u> Generalization : $S = \mathbb{Z}$

$m \sim n$ , $m - n$ is divisible by 3.

$$\mathcal{Cl}(3) = \mathcal{Cl}(0) = \{0, 3, -3, 6, -6, \dots\}$$
$$\mathcal{Cl}(1) = \{1, -2, 4, -5, \dots\}$$
$$\mathcal{Cl}(2) = \{2, -1, 5, -4, \dots\}.$$