Lecture: 04/17 Calculus lecture

Sec 9.8 Infinite series (continued)

Example from previous lecture.

Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n (x-4)^n}{n}$

Recall: Power series centered at $x=c$ has the form $\displaystyle\sum_{n=0}^{\infty} \underbrace{a_n (x-c)^n}_{u_n(x)} = \sum_{n=0}^{\infty} u_n(x)$

In this example: $c = 4$, $a_n = \dfrac{3^n}{n}$, and $u_n(x) = \dfrac{3^n (x-4)^n}{n}$

Step 1. Apply the Ratio test to $\displaystyle\sum_{n=0}^{\infty} u_n(x)$ and find the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{3^{n+1}(x-4)^{n+1}}{n+1}}{\dfrac{3^n (x-4)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}(x-4)\right|$$

$$= \lim_{n\to\infty}\left|\frac{3^{n+1}\cdot(x-4)^{n+1}}{3^n\cdot(x-4)^n}\,\frac{n}{n+1}\right| = \lim_{n\to\infty}\left|3(x-4)\cdot\frac{n}{n+1}\right| = \boxed{3|x-4|<1}$$

Series $\sum u_n(x)$ converges when

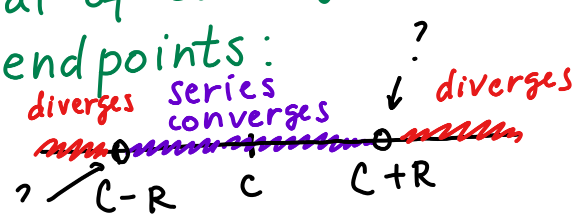
Solve for $|x-4|$ to find radius of convergence.

$$\Rightarrow |x-4| < \underset{R}{\underset{\shortparallel}{\tfrac{1}{3}}}\,.$$

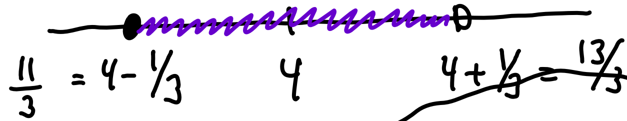
Radius of convergence $R = \frac{1}{3}$

Note: If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, then $R = \dfrac{1}{L}$.

Step 2 Draw the interval of convergence and investigate its endpoints: if $|x-c|<R$:



In the example: $|x-4| < 1/3$



Investigating endpoints:

When $x = 4 + 1/3$, $(x-4) = 1/3$, so $\displaystyle\sum_{n=0}^{\infty} \frac{3^n (x-4)^n}{n} = \sum_{n=0}^{\infty} \frac{3^n \cdot (1/3)^n}{n}$

$= \displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ diverges as p-series, $p = 1 \leq 1$

Thus $x = 13/3$ is <u>not</u> in the interval of convergence

When $x = 4 - 1/3$, $(x-4) = -1/3$, so $\displaystyle\sum_{n=0}^{\infty} \frac{3^n (x-4)^n}{n} = \sum_{n=0}^{\infty} \frac{3^n (-1/3)^n}{n}$

$= \displaystyle\sum_{n=0}^{\infty} \frac{3^n \cdot \frac{(-1)^n}{3^n}}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ ← converges by the Alternating Series test.

Thus $x = 11/3$ is in the interval of convergence

Answer: Radius of convergence: $R = 1/3$
Interval of convergence: $[11/3, 13/3)$

Note: convergence is absolute on $(11/3, 13/3)$

Example: (<u>Special cases</u>) Find radius and interval of convergence for the following series:

a) $\displaystyle\sum_{n=0}^{\infty} \underbrace{\frac{x^n}{n!}}_{u_n(x)}$ $(= e^x)$ ← always converges

Step 1. $\displaystyle\lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right|$

$= \displaystyle\lim_{n\to\infty} \left| \frac{x^{n+1} \cdot n!}{x^n \cdot (n+1)!} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 < 1$ for all x.

Answer: Radius of convergence: $R = +\infty$
Interval of convergence: $(-\infty, +\infty)$ (always converges!)

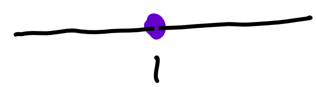
b) $\sum_{n=0}^{\infty} \underbrace{n!\,(x-1)^n}_{u_n(x)}$ ← converges when $x = 1$

Step 1. $\lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\,(x-1)^{n+1}}{n! \quad x^n} \right|$

$= \lim_{n\to\infty} \left| (n+1)(x-1) \right| = \begin{cases} 0, & x = 1 \\ +\infty, & x \neq 1. \end{cases}$

Answer. Interval of convergence is a single point 1

Radius of convergence: $R = 0$



Note: • Power series always converges absolutely inside its interval of convergence. (by Ratio test)

• At each endpoint of the interval of convergence, the series could converge absolutely, converge conditionally, or diverge. (by investigating endpoints)

• Power series always diverges outside its interval of convergence. (by Ratio test).

Differentiating and Integrating Power series.

If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ converges on interval (a,b), then we can differentiate and integrate $f(x)$ by differentiating and integrating power series. The resulting power series still converges on (a,b). Endpoints must be studied separately.

(I.e. integration or differentiation of power series preserves radius of convergence R.)

Example: Consider power series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n}$.

a) Find its radius and interval of convergence.

b) Differentiate the series and find its radius and interval of convergence.

c) Integrate the series and find its radius and interval of convergence.

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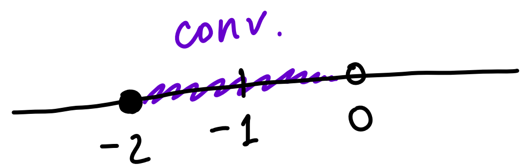
a) Step 1 $\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{n+1} \Big/ \frac{(x+1)^n}{n} \right| = \lim_{n\to\infty} \left| (x+1) \cdot \frac{n}{n+1} \right| = |(x+1)| < 1.$

$\boxed{\text{Radius of conv.: } R = 1}$



Step 2. Investigating endpoints:

- $x = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{(x+1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \leftarrow$ diverges
- $x = -2 \Rightarrow \sum_{n=1}^{\infty} \frac{(x+1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \leftarrow$ converges (conditionally)

$\boxed{\text{Interval of convergence: } [-2, 0)}$

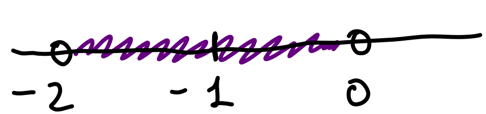
b) $\left(\sum_{n=1}^{\infty} \frac{(x+1)^n}{n} \right)' = \left(\frac{x+1}{1} + \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} + \dots + \frac{(x+1)^n}{n} + \dots \right)'$

$= \left(1 + (x+1) + (x+1)^2 + \dots + (x+1)^{n-1} + \dots \right)$

$= \sum_{n=0}^{\infty} (x+1)^n = a + ar + ar^2 + \dots$ $\boxed{R = 1}$

$\uparrow$ geometric $\quad r = x+1$

$|r| = |x+1| < 1$



Interval of conv. $(-2, 0)$

c) $\displaystyle\int\left(\sum_{n=1}^{\infty}\frac{(x+1)^n}{n}\right)dx = \int\left(\frac{(x+1)}{1}+\frac{(x+1)^2}{2}+\dots+\frac{(x+1)^n}{n}+\dots\right)dx$

$$= \frac{(x+1)^2}{2}+\frac{(x+1)^3}{2\cdot 3}+\frac{(x+1)^4}{3\cdot 4}+\dots+\frac{(x+1)^{n+1}}{n(n+1)}+\dots + C$$

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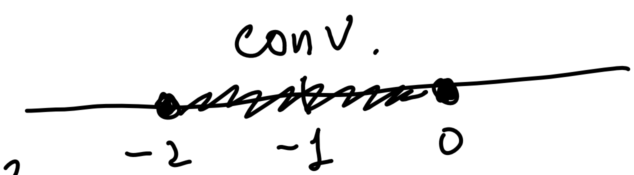
$$= \sum_{n=1}^{\infty}\frac{(x+1)^{n+1}}{n(n+1)}$$

Radius of conv. did not change.



Interval of convergence $[-2, 0]$

Step2

Investigating endpoints:

$x=-2$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}$

converges absolutely

$x=0$: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)} < \sum\frac{1}{n^2}$ converges

Solution to Quiz 22.

(a) Apply the Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}}\right| = \lim_{n\to\infty}\frac{n+1}{3} = \infty$$

Series diverges.

(b) Apply the Direct Comparison test:

$$\sum_{n=1}^{\infty}\frac{n^3}{n^5+n+1} < \sum_{n=1}^{\infty}\frac{n^3}{n^5} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$ p-series, $p=2>1$ converges.

So series $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{n^5+n+1}$ also converges.

(c) Divergence test: $\displaystyle\lim_{n\to\infty}(-1)^n\ln\left(\frac{3n}{2n+1}\right) = \lim_{n\to\infty}(-1)^n\ln\frac{3}{2}$, $=$ DNE so series diverges.

(d) Series $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ is geometric with $r = \pi/6 < 1$

$\Rightarrow$ converges.

Additional example.

Example. Find the intervals of convergence of (a) $f(x)$, (b) $f'(x)$, (c) $f''(x)$, (d) $\int f(x)\,dx$. Where

$$f(x) = \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n (x+3)^n}{n^2 \cdot 4^n}}_{u_n(x)}$$

Solution:

(a) Step 1 Apply the Ratio test to $\sum_{n=1}^{\infty} u_n(x)$:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x+3)^{n+1}}{(n+1)^2 \cdot 4^{n+1}} \Big/ \frac{(-1)^n (x+3)^n}{n^2 \cdot 4^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x+3)}{4} \cdot \frac{n^2}{(n+1)^2} \right| = \left| \frac{x+3}{4} \right| < 1 \Rightarrow$$

$$|x+3| < 4 \iff -4 < x+3 < 4 \iff -7 < x < 1$$

Radius of convergence $R = 4$

Series converges on $(-7, 1)$ and diverges on $(-\infty, -7) \cup (1, +\infty)$.

Step 2 Investigate endpoints of the interval of convergence.

If $x=-7$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(x+3)^n}{n^2\cdot 4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n(-7+3)^n}{n^2\cdot 4^n}=$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(-4)^n}{n^2\cdot 4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(-1)^n\cdot 4^n}{n^2\cdot 4^n}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

converges as p-series, $p=2>1$

If $x=1$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(1+3)^n}{n^2\cdot 4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ ← also converses, by Abs. convergence test, since $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Answer: $\boxed{\text{The interval of convergence is } [-7,1]}$

Note: f', f'', $\int f(x)dx$ will have the <u>same</u> radius of convergence (see p. 3). Behavior at endpoints could change.

b) $$f'(x)=\left(\sum_{n=1}^{\infty}\frac{(-1)^n(x+3)^n}{n^2\cdot 4^n}\right)'=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot n(x+3)^{n-1}}{n^2\cdot 4^n}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n(x+3)^{n-1}}{n\cdot 4^n}$$

Checking endpoints:

If $x=-7$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(-7+3)^{n-1}}{n\cdot 4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(-1)^{n-1}\cdot 4^{n-1}}{n\cdot 4^n}$

$$=\sum_{n=1}^{\infty}-\frac{1}{n\cdot 4}=-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n}$$ ← diverges, p series, $p=1$.

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If $x = 1$: $\sum_{n=1}^{\infty} \frac{(-1)^n (4)^{n-1}}{n \cdot 4^n} = \frac{1}{4} \sum_{n=1} \frac{(-1)^n}{n}$ ← converges, alt. series.

$\boxed{\text{Interval of convergence: } (-7, 1]}$

c) $f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n(n-1)(x+3)^{n-2}}{n^2 \cdot 4^n}$

If $x = -7$: $\sum_{n=1}^{\infty} \frac{(-1)^n n(n-1)(-4)^{n-2}}{n^2 \cdot 4^n} =$

$= \frac{1}{16} \sum_{n=1}^{\infty} \frac{n(n-1)}{n^2}$ ← diverges by divergence test. $\lim_{n\to\infty} \frac{n(n-1)}{n^2} = 1 \neq 0$

If $x = 1$: $\sum_{n=1}^{\infty} \frac{(-1)^n n(n-1) \cdot 4^{n-2}}{n^2 \cdot 4^n} = \frac{1}{16} \sum_{n=1}^{\infty} \frac{(-1)^n n(n-1)}{n^2}$

(also diverges, by divergence test).

Answer: $\boxed{\text{Interval of convergence } (-7, 1).}$

d) $\int f(x)\,dx = \int \left(\sum_{n=1}^{\infty} \frac{(-1)^n (x+3)^n}{n^2 \cdot 4^n} \right) dx$

$= \sum_{n=1}^{\infty} \frac{(-1)^n (x+3)^{n+1}}{(n+1) n^2 \cdot 4^n} + C$

If $x = -7$: $\sum_{n=1}^{\infty} \frac{(-1)^n (-4)^{n+1}}{(n+1) \cdot n^2 \cdot 4^n} = -4 \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}}$

direct comparison $\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^3}$ ← p-series, $p = 3$. converges by

If $x = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 4^{n+1}}{(n+1)n^2 \cdot 4^n} = \underbrace{4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2(n+1)}}_{\text{absolutely convergent, see above.}}$

<u>Answer</u>: Interval of convergence $[-7, 1]$.