Lecture: 04/14 Calculus lecture

Homework questions.

#30 p. 633 $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$. Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{((n+1)!)^2}{(3(n+1))!} \Big/ \frac{(n!)^2}{(3n)!}$$

$$= \lim_{n\to\infty} \frac{[(n+1)!]^2}{(n!)^2} \cdot \frac{(3n)!}{(3n+3)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(3n+1)(3n+2)(3n+3)} = 0$$

$(n+1)^2 = n^2+2n+1$

$(3n+1)(3n+2)(3n+3) = 27n^3 + \ldots$

$$\frac{(n+1)!}{n!} = \qquad \frac{(3n)!}{(3n+3)!} = \frac{1 \cdot 2 \cdot \ldots \cdot 3n}{1 \cdot 2 \cdot \ldots \cdot 3n \cdot (3n+1)(3n+2)(3n+3)}$$

Series converges.

#28 $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ Ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

$(n+1)^{n+1} = (n+1)^n \cdot (n+1)$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^n}$$

$$= \frac{1}{e} < 1$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a}$$

Series converges.

Review: Applying tests for convergence/divergence.

How to test series for convergence/divergence.

1. Does the n^{th} term approach 0? If not, the series diverges
2. Special cases: geometric, p-series, telescoping.
3. Can the Ratio test or the Root test be applied?
4. Is the series alternating?
5. Can the Limit Comparison or the Direct Comparison test be applied to compare series to a more simple one?
6. Can the Integral test be applied?

Last time: Example. Use any test to decide convergence/divergence.

c) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+1}} = \sum_{n=1}^{\infty} (-1)^n b_n$.

Use Alternating series test

$b_n = \frac{n}{\sqrt{n^3+1}} > 0$, decreasing

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^3+1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2(n+\frac{1}{n^2})}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1/n^2}} = 0$$

Series converges. (conditionally)

$f(x) = x + \frac{1}{x^2}$

$f'(x) = 1 - \frac{2}{x^3} > 0$

$x \geq 2$

d) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$ compare with $\sum_{n=1}^{\infty} \frac{n}{n^{3/2}}$

Diverges.

$= \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ ← diverges, p-seris $p = 1/2$.

Use limit comparison test.

e) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

$\sum_{n=1}^{\infty} (\ln(n+1) - \ln(n))$

$= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + \dots + (\ln(n+1) - \ln(n)) + \dots$

Telescoping

$S_k = -\ln 1 + \ln(k+1)$ (with $\ln 1 = 0$)

$\lim_{k\to\infty} S_k = \infty$

Series diverges

$\lim_{n\to\infty} \frac{n}{\sqrt{n^3+1}} \Big/ \frac{1}{n^{1/2}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+1}} = 1.$

f) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ ← diverges.

Direct or limit comparison tests do not work.

Integral test:

$\int_2^{\infty} \frac{1}{x \ln x}\,dx \overset{u=\ln x}{=} \infty$

g) $\sum_{n=1}^{\infty} \frac{4^n + 3^n}{n + 5^n} < \sum_{n=1}^{\infty} \frac{4^n + 4^n}{5^n} = 2 \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$

converges

geometric $r = 4/5 < 1$ converges.

Direct comparison

Limit comparison

Ratio test

h) $\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^n}$

↑ converges.

Use Ratio test:

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{e} < 1.$