Lecture: 04/13 Proof lecture

Comments about homework.

Prove that $\boxed{f(A \cap B) \subseteq f(A) \cap f(B)}$

Example exists that $f(A \cap B) \neq f(A) \cap f(B)$

$f(x) = x^2$, $A = \{1\}$, $B = \{-1\}$,

then $f(A \cap B) = \{\emptyset\}$, $f(A) \cap f(B) = \{1\}$

Notation issues: $\cancel{x \in \{f(x) \mid x \in A\}}$

$\cancel{x \in f(A)}$

$y = f(x) : A \to B$, $f(A) \subseteq B$

$y \in f(A) \iff \exists\, x \in A$ s.t. $f(x) = y$

$y \in f(A \cap B) \iff \exists\, x \in A \cap B$ s.t.

$f(x) = y \implies x \in A$ and $x \in B$ s.t.

$f(x) = y \implies y \in f(A)$ and $y \in f(B)$

$\not\Leftarrow$

$x_1 \in A \Rightarrow f(x_1) = y$

and $x_2 \in B \Rightarrow f(x_2) = y$

$\Rightarrow y \in f(A) \cap f(B)$

$\Leftarrow$

Functions (continued)

- $f: S \to T$ is one-to-one (injective) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in S$.

  (or $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$)

- $f: S \to T$ is onto (surjective) if for any $t \in T$ $\exists\, s \in S$ s.t. $f(s) = t$

  (or $f(S) = T$).

- $f: S \to T$ is bijective if it both one-to-one and onto.

Ex: How many bijective functions are from $S = \{1, 2, 3\}$ to $T = \{4, 5, 6\}$

$f: S \to T$ How many different f's?

↑ bijection.

$f(1) = 4 \to f(2) \to 5$, $f(3) = 6$

$ \to 6$, $ 5$

$f(1) = 5 \to 4$

$ \to 6$

$f(1) = 6 \to 4$

$ \to 5$ . . .

[3 choices for $f(1)$] · [2 choices for $f(2)$] · [1 choice for $f(3)$] $= 6$.

How many injective $f: S \to T$ are?

Also 6.

How many surjective $f: S \to T$ are?

Also 6.

$f(1) = 4$

$f(1) = 5$

$f(1) = 6$

$1 \to 5$

$2 \to 4$

$3 \to 5$

6

← not injective or surjective

Ex: $g : \{1, 2, 3\} \to \{4, 5\}$

a) How many surjective g?

<u>Exactly 6</u>

$\{1, 2\} \to \{4, 5\}$ exactly 2.

$$\left.\begin{array}{llll} & & & 3 \to 4 \\ & 1 \to 4 & 2 \to 5 & 3 \to 5 \\ & 1 \to 5 & 2 \to 4 & 3 \to 4 \\ & & & 3 \to 5 \end{array}\right\} \text{4 cases.}$$

$\{1, 3\} \to \{4, 5\}$ 4 cases

$\{2, 3\} \to \{4, 5\}$ 4 cases.

We double counting.

b) How many injective g?

None.

c) How many bijective g? None.

$h : \{1, 2, 3\} \to \{4, 5, 6, 7\}$

Inverse functions.

Definition: Let $f : S \to T$ be a bijection, then one can define an inverse function $g : T \to S$ as follows $g(t) = s$ if and only if $f(s) = t$.

Solving $\boxed{f(s) = t}$ for s, this solution exists since f is onto and the solution is unique, since f is one-to-one
$$g(t) = s.$$

Example: Let $f : [0,2] \to [0,4]$
$\boxed{f(x) = x^2}$ Define inverse function $g : [0,4] \to [0,2]$.

$g(3) = ?$ $f(x) = 3$, $x^2 = 3$,
$x_1 = \sqrt{3}$, ~~$x_2 = -\sqrt{3}$~~, so $g(3) = \sqrt{3}$.

You all know, $g(x) = \sqrt{x}$.

$g(y)$ is a unique solution of

$\underset{x^2}{\underset{\shortparallel}{f(x)}} = y$, $\quad y = \sqrt{x}$

<u>Def</u>: Let S, T, U be sets, $f : S \to T$, $g : T \to U$. Then their composition $h \cdot g \circ f : S \to U$ defined by $h(s) = g(\underbrace{f(s)}_{T})$ for all $s \in S$.

<u>Prop. 19.2</u> f, g are both one-to-one $\Rightarrow$ h is one-to-one

f, g are both onto $\Rightarrow$ h also onto

f, g are both bijections $\Rightarrow$ h is a bijection.