Lecture: Lecture on 04/13/2020

Solutions to Quiz #20

1(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Apply alt. series test. $b_n = \frac{1}{\sqrt{n}} > 0$, decreasing, $\lim_{n\to\infty} b_n = 0 \Rightarrow$ series converges.

Checking absolute convergence. $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges as p-series, $p = 1/2$

<u>Answer:</u> $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges conditionally.

(b) $5 - 5/2 + 5/2^2 - 5/2^3 + \ldots$ is geometric, $a = 5$, $r = -1/2$, $|r| = 1/2 < 1 \Rightarrow$ series converges.

(One can also apply Alt. series test with $b_n = \frac{5}{2^{n-1}}$).

Checking abs. convergence: $|5| + |-5/2| + |5/2^2| + |-5/2^3| + \ldots$

$= 5 + 5/2 + 5/2^2 + \ldots$ is also geometric, $a = 5$, $r = 1/2 < 1$

$\Rightarrow$ series converges.

<u>Answer:</u> Series $5 - 5/2 + 5/2^2 - 5/2^3 + \ldots$ converges absolutely.

(c) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{2^n + n}$ $\quad \lim_{n\to\infty} \underbrace{(-1)^{n+1} \frac{2^n}{2^n+n}}_{a_n} = \lim_{n\to\infty} (-1)^{n+1} \underbrace{\frac{\cancel{2^n}}{\cancel{2^n}(1 + \frac{n}{2^n})}}_{=1}$

$= \lim_{n\to\infty} (-1)^{n+1} = \text{DNE}$

<u>Answer.</u> Series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{2^n+n}$ diverges by divergence test.

Sec. 9.6. Ratio and root test (continued):

Thm (Ratio test) Let $\sum_{n=1}^{\infty} a_n$ be a series with non-zero terms. Assume that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$$

Then the series

$$\sum_{n=1}^{\infty} a_n \begin{cases} \text{converges absolutely}, & \text{if } r<1 \\ \text{diverges}, & \text{if } r>1, \text{ or } \infty \\ \text{test inconclusive}, & \text{if } r=1. \end{cases}$$

Remark: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist or if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then Ratio test is inconclusive, apply some other test.

Thm (Root test): Let $\sum_{n=1}^{\infty} a_n$ be any series and suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists and $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

then

$$\sum_{n=1}^{\infty} a_n \begin{cases} \text{converges absolutely}, & \text{if } L<1, \\ \text{diverges}, & \text{if } L>1, \text{ or } \infty \\ \text{test inconclusive}, & \text{if } L=1. \end{cases}$$

Remark: If $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ does not exist, or if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then Root test is inconclusive, apply other tests.

Examples: Apply root test (if possible) to decide convergence / divergence.

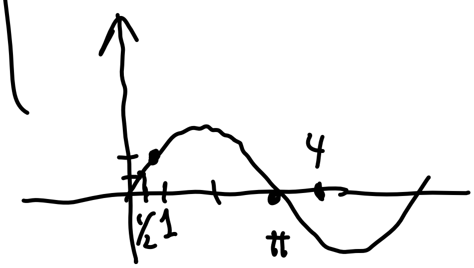
a) $\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n}_{\text{geometric.}}$, $\quad \lim_{n\to\infty} \underset{\sqrt[n]{|a_n|}}{\underset{\shortparallel}{(|a_n|)^{1/n}}} = \lim_{n\to\infty}\left(\left(\frac{1}{2}\right)^n\right)^{1/n} = \frac{1}{2} < 1$

$\Rightarrow$ series converges.

b) $\sum_{n=1}^{\infty}\left(\sin\left(\frac{1}{n}\right)\right)^n$, $\quad \underbrace{\lim_{n\to\infty}(|a_n|)^{1/n}} = \lim_{n\to\infty}\left(\left(\sin\left(\frac{1}{n}\right)\right)^{\cancel{n}}\right)^{\cancel{1/n}}$

$= \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin 0 = 0 < 1$

$\sin(1), \sin(\frac{1}{2}), \sin(1/3), \ldots$



Series converges.

c) $\sum_{n=1}^{\infty}\frac{n^n}{100^n}$, $\quad \lim_{n\to\infty}\left(\frac{n^{\cancel{n}}}{100^{\cancel{n}}}\right)^{\cancel{1/n}} =$

$= \lim_{n\to\infty}\frac{n}{100} = \infty > 1$, series diverges

d) $\sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{n}\right)^{n^2}$

$\lim_{n\to\infty}\left|\cancel{(-1)^n}\underbrace{\left(1-1/n\right)}^{n^2}\right|^{1/n}$

$= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x \overset{1^\infty}{=}$

$= \lim_{x\to\infty} e^{\ln\left(1-\frac{1}{x}\right)^x} = e^{\lim_{x\to\infty} x\ln(1-1/x)}$

$= e^{\lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{1/x}} \overset{0/0}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{1-1/x}\cdot\cancel{(1/x^2)}}{-\cancel{(1/x^2)}}}$

$= e^{-1} = \frac{1}{e} < 1$ Series converges absolutely

$\boxed{\left(1+\frac{a}{x}\right)^x \xrightarrow[x\to\infty]{} e^a}$

How to test series for convergence/divergence.

1. Does the n^{th} term approach 0? ($\lim_{n\to\infty} a_n = 0$?)
   If not, the series diverges.
2. Special cases: geometric, p-series, telescoping
3. Can the Ratio test or the Root test be applied.
4. Is the series alternating?
5. Can the series be compared favorably to one of the special types by applying Limit Comparison or Direct Comparison tests.
6. Can Integral test be applied?

Nothing works: $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ ← positive or negative

$\sin(1), \sin(2), \sin(3)$ all pos. $\sin(4), \sin(5), \sin(6)$ are neg.

Ex: Decide convergence/divergence.

a) $\displaystyle\sum_{n=1}^{\infty} \underbrace{(-1)^n \cos(1/n)}$

$$\lim_{n\to\infty} (-1)^n \underbrace{\cos(1/n)}_{\downarrow\ \cos(0)=1} = \lim_{n\to\infty} (-1)^n \cdot 1 = \text{DNE} \neq 0$$

Series diverges by divergence test.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n!}$

$$\frac{5\cdot 5\cdot 5\cdot \ldots \cdot 5 \cdot 5 \cdot 5}{1\cdot 2\cdot 3\cdot \ldots \cdot (n-2)(n-1)\, n} \xrightarrow[n\to\infty]{} 0$$

Apply Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n}$$

$$= \lim_{n\to\infty} \frac{5}{n+1} = 0 < 1$$

Series converges.

c) $\displaystyle\sum_{n=1}^{\infty} \underbrace{(-1)^n} \frac{n}{\underbrace{\sqrt{n^3+1}}_{\text{like } n^{3/2}}}$

(Ratio or root test: No! Does not work well for powers)

Apply Alt. series test.