Lecture: 04/09 Calculus lecture.

Comments on quiz # 19.

- How to apply limit comparison test to investigate the series $\sum_{n=1}^{\infty} a_n$.

Step 1. Verify that $a_n \geq 0$ for all n.

Step 2 Based on the formula for a_n choose the comparison series $\sum_{n=1}^{\infty} b_n$, that simplifies $\sum_{n=1}^{\infty} a_n$. Good choices for $\sum_{n=1}^{\infty} b_n$ are p-series, or geometric series.

Step 3. Verify that $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$.

Step 4. Decide if the series $\sum_{n=1}^{\infty} b_n$ converges or diverges. Same conclusion applies to $\sum_{n=1}^{\infty} a_n$.

Example from quiz 19: $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+5}}{n - \sqrt{n+5}} \right) = \sum_{n=1}^{\infty} a_n$

Step2. Compare to $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Step 3. $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ (see quiz solutions)

Step 4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges as p-series, $p = \frac{1}{2} < 1$.

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n+5}}{n - \sqrt{n+5}}$ also diverges.

Homework questions.

Mistake 1. Not comparing to anything.

Mistake 2. Comparing, but not investigating b_n.

Mistake 3. Comparing to the wrong b_n.

Compare to $\sum_{n=1}^{\infty} \frac{\sqrt{n+5}}{n}$ converges or diverges.

$\int_1^{\infty} \frac{\sqrt{x+5}}{x}\,dx \leftarrow$ finite or infinite.

Abs. convergence $\Rightarrow$ convergence

\# 49 $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$ Alt. series. Does it converge or diverges.

converges absolutely

$= \sum_{n=0}^{\infty} (-1)^n b_n$, $b_n = \frac{1}{(2n+1)!} = \frac{1}{1\cdot 2\cdot 3\cdot 4 \cdots 2n\cdot(2n+1)}$

$b_{n+1} < b_n$, $\lim_{n\to\infty} b_n = 0 \Rightarrow$ series converges by Alt. series test.

Abs. convergence $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{(2n+1)!}\right| = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$

not geom.

- $\sum ar^{n-1} = a + ar + ar^2 + \ldots$ r is a number
- $\sum \frac{1}{n^p}$ Direct comparison $\sum \frac{1}{(2n+1)!} < \sum \frac{1}{2n(2n+1)} < \sum \frac{1}{4n^2}$

$p = 2$, converges

Sec. 9.6. Ratio and root test.

- Recall geometric series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} \text{converges to } \frac{a}{1-r}, \text{ if } |r|<1 \\ \text{diverges, if } |r| \geq 1. \end{cases}$$

Notice that for this series

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{a} r^{n+1-1}}{\cancel{a} r^{n-1}}\right| = |r|$$

So for geometric series $\sum_{n=1}^{\infty} a_n = \begin{cases} \text{converges, if } \left|\frac{a_{n+1}}{a_n}\right|<1 \\ \text{diverges, if } \left|\frac{a_{n+1}}{a_n}\right| \geq 1 \end{cases}$

- Now consider the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ ← not geometric, but looks similar.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{2^{n+1}} \Big/ \frac{n}{2^n}\right| = \left|\frac{n+1}{n} \cdot \frac{\cancel{2^n}}{\cancel{2}^{n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}.$$

For large values of n, $\sum_{n=1}^{\infty} \frac{n}{2^n}$ behaves like a geometric series with $r = 1/2$, so we expect this series to converge.

Theorem (Ratio test) Let $\sum_{n=1}^{\infty} a_n$ be a series with non-zero terms. Assume that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r.$$

Then

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \text{converges absolutely}, & \text{if } r<1; \\ \text{diverges}, & \text{if } r>1; \text{ or if } r=\infty. \\ \text{test inconclusive}, & \text{if } r=1. \end{cases}$$

Remark: 1) Ratio test compares the behavior of $\sum_{n=1}^{\infty} a_n$ to the behavior of geom. series $\sum_{n=1}^{\infty} ar^{n-1}$ for large n.

2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ or DNE, then ratio test is not applicable.

3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

very fast

Examples:

a) $\sum_{n=1}^{\infty} \underbrace{\frac{n^2 \cdot 2^n}{3^n}}_{a_n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^{n+1}}{3^{n+1}} \Big/ \frac{n^2 \cdot 2^n}{3^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n^2 \cdot 2^n}$$

$$= \lim_{n\to\infty} \boxed{\frac{(n+1)^2}{n^2}} \cdot \frac{2}{3} = 2/3 < 1.$$

So the series converges.

$$\left(\frac{n+1}{n}\right)^2 = \left(\frac{\cancel{n}(1+1/n)}{\cancel{n}}\right)^2 = \left(1+\frac{1}{n}\right)^2 \xrightarrow{n\to\infty} 1$$

b) $\sum_{n=1}^{\infty} \underbrace{(-1)^n \cdot \frac{3^n}{n!}}_{a_n}$ ← absolutely converges.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot 3^{n+1}}{(n+1)!} \Big/ \frac{(-1)^n \cdot 3^n}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}}{\cancel{(-1)^n}} \cdot \frac{n!}{(n+1)!} \cdot \frac{3^{n+1}}{3^n} \right|$$

$$= \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{3}{1} = 0 < 1$$

$$\frac{n!}{(n+1)!} = \frac{\cancel{[1\cdot 2\cdot 3 \cdots (n-1)\, n]}}{\cancel{[1\cdot 2\cdot 3 \cdots (n-1)\, n]}\,(n+1)} = \frac{1}{n+1}$$

c) $\sum_{n=1}^{\infty} \frac{(2n)!}{100^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\overbrace{(2(n+1))!}^{(2n+2)!}}{100^{n+1}} \Big/ \frac{(2n)!}{100^n}$$

$$= \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \cdot \frac{\cancel{100^n}}{100^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{100} = \infty > 1 \text{. diverges.}$$

$$\frac{(2n+2)!}{(2n)!} = \frac{\cancel{1\cdot 2\cdot 3\cdot \ldots (2n)}\cdot(2n+1)(2n+2)}{\cancel{1\cdot 2\cdot 3\cdot \ldots (2n)}}$$

Series diverges.

d)? $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ ← diverges.

compare $\sum \frac{n}{n^2} = \sum \frac{1}{n}$

limit comparison test

Apply ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{(n+1)^2+1} \Big/ \frac{n}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{(n^2+1)(n+1)}{((n+1)^2+1)\cdot n} =$$

Test does not apply.

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{1}{n^2}\right)\cdot \cancel{n}\left(1+1/n\right)}{\left(\cancel{n^2}\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right)\cdot \cancel{n}} = 1.$$

Ratio test is <u>not</u> applicable to the ratio of two polynomials.

#46

$$\sum_{n=0}^{\infty} (-1)^n \underbrace{e^{-n^2}}_{b_n}$$ ← apply alt. series test

$\lim_{n\to\infty} b_n = 0$, $b_{n+1} < b_n$

series converges.

abs. converges?

Abs. convergence:

$$\sum_{n=1}^{\infty} \left| (-1)^n e^{-n^2} \right| =$$

$$= \sum_{n=1}^{\infty} e^{-n^2} = \sum_{n=1}^{\infty} \frac{1}{e^{n^2}} < \underbrace{\sum_{n=1}^{\infty} \frac{1}{e^n}} = \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n}$$

converges.

$n^2 > n$

$e^{n^2} > e^n \Rightarrow \frac{1}{e^{n^2}} < \frac{1}{e^n}$

geom.

$r = \frac{1}{e} < 1$.

converges.