Lecture: 04/08 Proof lecture.

Comments on hwk #16

- Redo the homework #16 due Monday, April 13 before class

Example: Let $A_n = \{x \in \mathbb{R} \mid -n < x < \frac{1}{n}\}$. Find $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ and prove your answers.

Solution: • $\bigcup_{n=1}^{\infty} A_n = (-\infty, 1)$

Show that $\bigcup_{n=1}^{\infty} A_n \subseteq (-\infty, 1)$: Let $x \in \bigcup_{n=1}^{\infty} A_n \Rightarrow \exists m$ s.t. $x \in A_m \Rightarrow -m < x < \frac{1}{m} \Rightarrow x \in (-\infty, 1)$.

Show that $(-\infty, 1) \subseteq \bigcup_{n=1}^{\infty} A_n$: Let $x \in (-\infty, 1)$.

If $x \geq 0$, then $x \in A_1 = (-1, 1) \Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$.

If $x < 0$, then by Arch. principle $\exists m \in \mathbb{N}$ s.t. $-x < m$ or $x > -m \Rightarrow x \in A_m = (-m, \frac{1}{m})$
$\Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$

• $\bigcap_{n=1}^{\infty} A_n = (-1, 0]$

Show that $\bigcap_{n=1}^{\infty} A_n \subseteq (-1, 0]$ Suppose not, then there is $x \in \bigcap_{n=1}^{\infty} A_n$ and $x \notin (-1, 0]$.

If $x > 0$, then by Arch. principle $\exists\, m \in \mathbb{N}$ s.t. $0 < \frac{1}{m} < x \Rightarrow x \notin A_m = (-1/m, m) \Rightarrow x \notin \bigcap_{n=1}^{\infty} A_n$. So this case is not possible.

If $x \leq -1$, then $x \notin A_1 = (-1, 1) \Rightarrow x \notin \bigcap_{n=1}^{\infty} A_n$.

So this case is also not possible.

Thus, if $x \in \bigcap_{n=1}^{\infty} A_n$, then $x \in (-1, 0)$.

Show that $(-1, 0] \subseteq \bigcap_{n=1}^{\infty} A_n$. If $x \in (-1, 0]$, then $x \in A_n = (-n, 1/n)$ for all $n \Rightarrow x \in \bigcap_{n=1}^{\infty} A_n$

In problem #2 in hwk #16 you need to

- Define set A_n, A_n is infinite.
- Show that $A_i \cap A_j = \emptyset$ for all $i \neq j$.
- Show that $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$

Hwk #17 questions.

#1 c) $\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} (A_i^c)$

Why can't one use induction?

$x \in \left(\bigcup_{i=1}^{\infty} A_i \right)^c \Leftrightarrow x \notin \bigcup_{i=1}^{\infty} A_i \Leftrightarrow$

$\underbrace{x \notin A_1}_{x \in A_1^c}$ and $x \notin A_2$ and ... and $x \notin A_i$, ... $\iff x \in A_1^c$ and $x \in A_2^c$ ...

$\iff$

Chapter 19. Functions (continued)

Function $f: S \to T$ is a rule s.t. $\forall s \in S$ $\exists$ unique t in T denoted $f(s) = t$.

S = domain of f

Range of f: $f(S) \subseteq T$, $f(S) = \{ f(s) \mid s \in S \}$

Examples of functions:

a) $\ell : P_n(\mathbb{R}) \to \mathbb{R}$ $\quad \ell(p(x)) = \int_0^1 p(x)\,dx$

$P_n(\mathbb{R})$: all polynomials of degree $\leq n$ with real coeff.

$\boxed{n=2}$

Ex: $\ell(1) = \int_0^1 1\,dx = 1$

$$\ell(x^2+x+2) = \int_0^1 (x^2+x+2)\,dx = \frac{1}{3} + \frac{1}{2} + 2 = \frac{17}{6}$$

Domain: $P_n(\mathbb{R})$
Range: $\mathbb{R}$.

b) $D: P_n(\mathbb{R}) \to P_n(\mathbb{R})$

$$D(p(x)) = p'(x)$$
$$D(1) = (1)' = 0$$
$$D(x^2+x+2) = (x^2+x+2)' = 2x+1.$$

Domain: $P_n(\mathbb{R})$
Range: $P_{n-1}(\mathbb{R})$

Def: • A function $f: S \to T$ is called one-to-one (or injective) if for all $x, y \in S$, $f(x) = f(y) \Rightarrow x = y$.

(i.e. $x \neq y \Rightarrow f(x) \neq f(y)$)

[Each element in the range came from exactly one element in the domain]

• A function $f: S \to T$ is called onto (or surjective) if for each $t \in T$, there is $s \in S$ s.t. $f(s) = t$.

$(f(S) = T)$

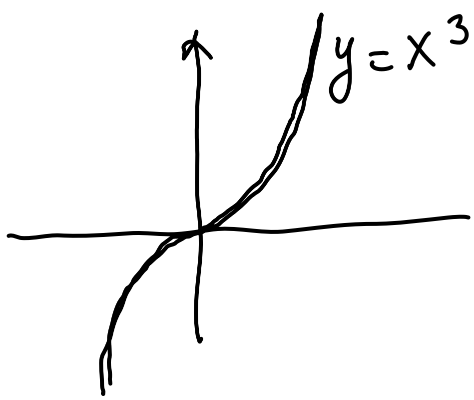
. A function $f: S \to T$ is called <u>bijective</u> if it is both one-to-one and onto.

<u>Ex:</u> 1) $g(x): \mathbb{R} \to \mathbb{R}$, $g(x) = x^3$

Is g one-to-one?

- Let $g(x_1) = g(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow (x_1^3)^{1/3} = (x_2^3)^{1/3}$
$\Rightarrow x_1 = x_2$.

- <u>OR</u> $x_1 \neq x_2$, say $x_1 < x_2 \Rightarrow$
$x_1^3 < x_2^3 \Rightarrow f(x_1) \neq f(x_2)$



strictly Increasing $\Rightarrow$ injective.

Is g onto? Given any $\underline{b} \in \mathbb{R}$,
need $a \in \mathbb{R}$ so that $g(a) = b$.
Take $a = (b)^{1/3}$: $g(b^{1/3}) = (b^{1/3})^3 = b$.
g is a bijection

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2) $h(x): \mathbb{R} \to \mathbb{R}$, $h(x) = x^2$

Is h one-to-one? No.
$h(1) = h(-1)$, but $1 \neq -1$

Is h onto? $h(a) = -1$ or $a^2 = -1$ has no real solutions, so h is not onto.

3) $f : [0, +\infty) \longrightarrow [0, +\infty)$

$f(x) = x^2$ ← this function is both one-to-one and onto.

$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = x_2$ or $\boxed{x_1 = -x_2}$ not possible unless $x_1 = -x_2 = 0$

4) Injective function

$\ell : [0,2] \to [0,1]$ which is not surjective.

Let $\ell(x) = \dfrac{x}{4}$ ← one-to-one.

$\dfrac{x_1}{4} = \dfrac{x_2}{4} \Rightarrow x_1 = x_2.$

Range of $\ell(x)$? $[0, \frac{1}{2}]$, not onto