Lecture: Calculus II lecture on 04/07/20.

Solutions to quiz 19.

Apply limit comparison test

1a) $\sum_{n=1}^{\infty}\left(\frac{\sqrt{n+5}}{n-\sqrt{n}+5}\right) = a_n$ will compare to $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, b_n

$= \lim a_n/b_n$

$$\lim_{n\to\infty}\left(\frac{\sqrt{n+5}}{n-\sqrt{n}+5}\right)\Big/\left(\frac{1}{\sqrt{n}}\right) = \lim_{n\to\infty}\frac{\sqrt{n}\cdot\sqrt{n(1+5/n)}}{n(1-\frac{\sqrt{n}}{n}+5/n)} =$$

$$= \lim_{n\to\infty}\frac{\cancel{\sqrt{n}\cdot\sqrt{n}}\cdot\sqrt{1+5/n}}{\cancel{n}(1-\frac{1}{\sqrt{n}}+5/n)} = 1/1 = 1 > 0$$

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges as a p-series with $p = 1/2$,

so does $\sum_{n=1}^{\infty}\frac{\sqrt{n+5}}{n-\sqrt{n}+5}$

b) $\sum_{n=1}^{\infty}\frac{3^n-2^n}{3^n+4^n}$ will compare to $\sum_{n=1}^{\infty}\frac{3^n}{4^n}$.

$$\lim_{n\to\infty}\left(\frac{3^n-2^n}{3^n+4^n}\right)\Big/\left(\frac{3^n}{4^n}\right) = \lim_{n\to\infty}\frac{\cancel{4^n}\cdot(\cancel{3^n}(1-(\frac{2}{3})^n))}{\cancel{3^n}\cdot(\cancel{4^n}((\frac{3}{4})^n+1)}$$

$= 1 > 0$.

Since $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ is geom. with $r = 3/4$, it converges.

Then $\sum_{n=1}^{\infty}\frac{3^n-2^n}{3^n+4^n}$ also converges

Homework questions.

80. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ Direct comparison

Integral test $\int_1^{\infty} \frac{\ln x}{x}\,dx$

$\frac{1}{n} < \frac{\ln n}{n}$, $n > 3$

$\sum_{n=3}^{\infty} \frac{1}{n} = \infty \overset{<}{\Rightarrow} \sum_{n=3}^{\infty} \frac{\ln n}{n} = \infty$

↑ diverges.

#6 $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1} n}{3n+2}}_{a_n}$ ← diverges by divergence test.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1} \cdot \cancel{n}}{\cancel{n}(3 + 2/n)}$$

$$= \lim_{n\to\infty} \frac{(-1)^{n+1}}{3} = \text{DNE}$$

Sec. 9.5. Alternating series (continued).

Recall Alternating series test: Suppose in $\sum_{n=1}^{\infty}(-1)^n b_n$ or in $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$ all b_n's are positive, b_n's are decreasing and $\lim_{n\to\infty} b_n = 0$. Then series converges.

Note: • In order to apply the test one needs to check that
- $b_n \geq 0$ for all n
- $b_{n+1} \leq b_n$ for all n
- $\lim_{n\to\infty} b_n = 0$

• Alternating series test is a test of convergence. So the only possible answers for this test are: series converges or the test is not applicable (need to apply some other test).

Absolute and conditional convergence.

Ex: Consider the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \dots$

$\sin 1 \approx .84$, $\sin 2 \approx .91$, $\sin 3 \approx .14$, $\sin 4 \approx -.76$, $\sin 5 \approx -.96$

This series has positive and negative terms, but it is not an alternating series, since the signs do not follow $+, -, +, - \dots$ pattern.

- All the tests we know so far fail to apply: this series is not alternating; it has infinitely many negative and positive terms, so direct/limit comparison tests also fail to apply.

- On the other hand, the series $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ has non-neg. terms. Apply Direct comparison test.

$0 \le \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$, $\sum \frac{|\sin n|}{n^2} < \sum \frac{1}{n^2}$ ← converges p-series $p = 2 > 1$.

↑ converges.

Absolute convergence test. If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

Ex: So the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges since $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ converges.

Explanation of absolute convergence test. Suppose $\sum |a_n|$ converges.

Write $a_n = a_n + |a_n| - |a_n|$.

$5 + |5| = 2 \cdot 5 = 2|5|$

$-5 + |-5| = -5 + 5 = 0$

Observe that $0 \le a_n + |a_n| \le 2|a_n|$, so the series $\sum_{n=1}^{\infty} a_n + |a_n|$ converges by Direct comparison test.

Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$ also converges as a difference of two convergent series.

- A series $\sum_{n=1}^{\infty} a_n$ is called <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_n|$ converges (then $\sum_{n=1}^{\infty} a_n$ also converges)
- If $\sum_{n=1}^{\infty} a_n$ converges, but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ is called <u>conditionally convergent</u>.

<u>Examples:</u> Is each series below convergent or divergent? For convergent series, decide absolute/conditional convergence.

1. $\sum_{n=1}^{\infty} (-1)^n \underbrace{1/n}_{=b_n}$ ← alternating series
$= -1 + 1/2 - 1/3 + \cdots$ ↑ converges by alt. series test

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \leftarrow \text{diverges.}$$

$|(-1)^n| = |\pm 1| = 1.$

So $\sum_{n=1}^{\infty} (-1)^n \cdot 1/n$ converges conditionally.

2. $\sum_{n=1}^{\infty} (-1)^n \, 1/n^2$ ← converges by Alt. series test.

$$\sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.

3. $+1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \dots$ ← also conv. absolutely

Not alternating. Not geometric.

$a + ar + ar^2 + \dots$

Use absolute convergence test:

$|1| + |\frac{1}{2}| + |-\frac{1}{4}| + |\frac{1}{8}| + |\frac{1}{16}| + |-\frac{1}{32}| + \dots$

$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$

geometric, $r = \frac{1}{2}$, so it converges.

So the original series also converges, because of abs. conv. test.

4. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\ln n + 1}$

Can't apply alt series test

$\lim_{n \to \infty} \frac{\ln n}{\ln n + 1} = 1$, $\lim_{n \to \infty} (-1)^n \underbrace{\frac{\ln n}{\ln n + 1}}_{\to 1} = \text{DNE}$

diverges by divergence test.

(not absolutely or conditionally convergent)

5. $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n}) = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n+1} + \sqrt{n}}$

Algebra: $(\sqrt{n+1} - \sqrt{n}) = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$ $= 1$

Apply alt. series test: $b_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} > 0$,

decreasing, $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}+\sqrt{n}}$ converges.

$$\sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{1}{\sqrt{n+1}+\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$ compare to $\sum \frac{1}{2\sqrt{n}}$

diverges.

Apply limit comparison test.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges.

Original series converges conditionally.