Lecture: Calculus lecture on 04/06.

Solutions to quiz #18

1a) Since $\frac{n^2}{n^4+1} < \frac{n^2}{n^4} = \frac{1}{n^2}$, we have that $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$. Series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p = 2 > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$ also converges by the direct comparison test.

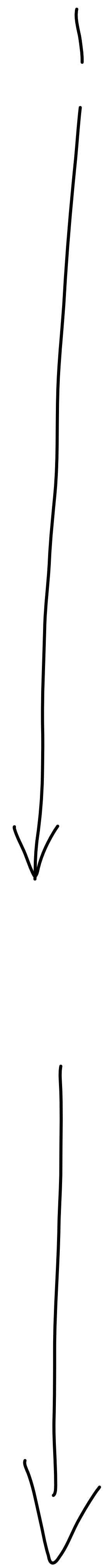
b) Since $\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}-1}$, we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$. Series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = \frac{1}{2} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$ also diverges by the direct comparison test.

Homework questions.

#20 p. 616 $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}} \leftarrow$ converges.

compare to $\sum_{n=1}^{\infty} \frac{\cancel{n}}{\cancel{n} \cdot 2^{n-1}} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}}_{\text{convergent geom. } r = \frac{1}{2} < 1.}$

$$\lim_{n \to \infty} \frac{\frac{n}{(n+1)\cancel{2^{n-1}}}}{\frac{1}{\cancel{2^{n-1}}}} = 1 > 0$$



Two quick examples. 1) $\lim_{n\to\infty} \frac{1/n!}{1/n^2} = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n!} =$

$\lim_{n\to\infty} \frac{n^2}{1\cdot 2\cdot 3\cdots(n-2)(n-1)n} = 0$

and we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, what $= \sum b_n$

can we say about $\sum_{n=1}^{\infty} \frac{1}{n!} = \sum a_n$

2) $\lim_{n\to\infty} \frac{1/\ln n}{1/n} = \lim_{n\to\infty} n/\ln n = \infty$

and we know that $\sum_{n=2}^{\infty} \frac{1}{n} = b_n$ diverges, what can we say about $\sum_{n=2}^{\infty} \frac{1}{\ln n} = a_n$?

smaller

bigger also have to diverge.

Extension of limit comparison test. Let $a_n > 0$ and $b_n > 0$ for all n.

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Sec. 9.5 Alternating series (handout)

- Let $b_1, b_2, \dots, b_n \dots$ be all positive, an alternating series is the series of the form

$$\sum_{n=1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + \dots + (-1)^n b_n + \dots$$

or

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \dots + (-1)^{n+1} b_n + \dots$$

Examples of alternating series:

a) $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = -\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}-\frac{1}{2^5}+\cdots$

b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ ← converges.

$\sum \frac{1}{n}$ diverges

Alternating series test: Suppose $b_1, b_2, \ldots, b_n, \ldots$ are all positive and form a decreasing sequence (i.e. $b_{n+1} \le b_n$ for all n) and $\lim_{n\to\infty} b_n = 0$, then the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges or $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$ converges.

Examples:

(a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \sum_{n=1}^{\infty}(-1)^{n+1} b_n$

$b_n = \frac{1}{n} > 0$ decreasing $\lim_{n\to\infty}\frac{1}{n}=0$

apply alt. series test $\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges.

(Do not confuse with $\sum \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\cdots$ not alternating, it diverging p-series)

(b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p}$, $b_n = \frac{1}{n^p}$ $p>0$, then $\frac{1}{n^p}$ is decreasing, $\lim_{n\to\infty}\frac{1}{n^p}=0$

Apply alt. series test

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} \text{ converges:}$$

$$\left.\begin{array}{l} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \; p=2 \\ \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}, \; p=\tfrac{1}{2} \\ \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}, \; p=\tfrac{1}{3} \end{array}\right\} \text{converge by alt. series test}$$

Not alt. series

$\sum \frac{1}{n^2}$ conv.

$$\left.\begin{array}{l} \sum \frac{1}{\sqrt{n}} \\ \sum \frac{1}{\sqrt[3]{n}} \end{array}\right\} \text{div.}$$

c) $$\underbrace{\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}}_{\text{alternating series}}$$

↑ diverges by divergence test.

Apply divergence test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{n}{n+1} \quad \frac{n}{n+1} \to 1$$

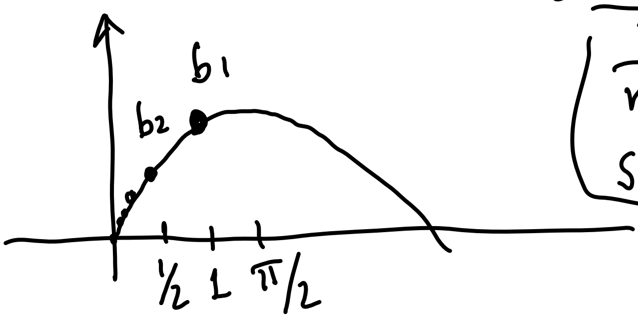
$$= \lim_{n\to\infty} (-1)^n = \text{DNE}$$

↑ oscillates.

<u>Important</u>: Alternating series test cannot prove divergence, it can only prove convergence.

Ex: a) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$, $b_n = \sin\left(\frac{1}{n}\right)$

b_n is decreasing:



$$\frac{1}{n+1} < \frac{1}{n}$$

$$\sin\left(\frac{1}{n+1}\right) < \sin\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0$$

By alt. series test $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ is convergent.

b) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$

diverges by divergence test

$$\lim_{n\to\infty} (-1)^n \cos\left(\frac{1}{n}\right) = \lim_{n\to\infty} (-1)^n = \text{DNE}.$$

c) $\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n$

$\to$ alt. series test $b_n = \left(\frac{2}{3}\right)^n$

$\to$ geometric series

converges.