Lecture: 04/03 Calculus lecture

Solutions to quiz #17.

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$$

1(a) Integral test. $f(x) = \frac{1}{x^2 \ln x} > 0$, continuous, decr. on $[2, +\infty)$.

$$\int_2^{\infty} \frac{1}{x^2 \ln x}\,dx = \lim_{t \to \infty} \int_2^t \frac{1}{x^2 \ln x}\,dx < \lim_{t\to\infty} \int_2^t \frac{2}{x^2}\,dx$$

$\frac{1}{\ln x} \le \frac{1}{\ln 2} < 2$

$$= \lim_{t\to\infty} -\frac{2}{x}\Big|_2^t = \lim_{t\to\infty}\left(-\frac{2}{t} + 1\right) = 1.$$

Series converges.

(b) $3 - 3/5 + 3/5^2 - \ldots = a + ar + \ldots$ is geom. with $a = 3$, $r = -1/5$. Series converges to

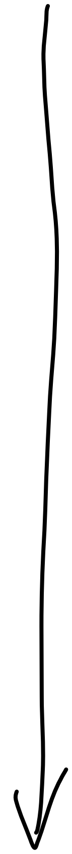
$$\frac{a}{1-r} = \frac{3}{1+1/5} = \underline{5/2}.$$

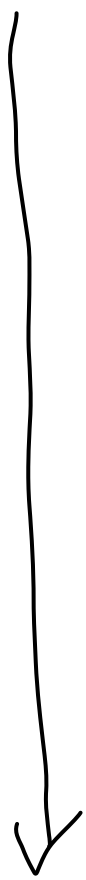
(c) $\sum_{n=1}^{\infty} \frac{n+1}{n+1000}$ Divergence test: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n+1000} =$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}(1+\frac{1000}{n})} = 1 \neq 0.$$

Series diverges

Homework questions.





Sec. 9.4. Comparison of series (continued)

Limit comparison test (theorem 8.13):

Suppose $a_n > 0$ and $b_n > 0$ for all n and $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L > 0$. ($L \neq \infty$) Then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

Idea: If $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L > 0$, then sequences (a_n) and (b_n) have similar behavior as $n \to \infty$.

Q1: What do you think happens if $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 0$?

Q2: What do you think happens if $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \infty$?

Ex 1: a) $\sum_{n=2}^{\infty} \underbrace{\frac{1}{n^3-7n+6}}_{a_n}$ compare to $\sum_{n=2}^{\infty} \underbrace{\frac{1}{n^3}}_{b_n}$ ← converges p-series $p = 3 > 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^3-7n+6}}{1/n^3} =$$

$$= \lim_{n\to\infty} \frac{n^3}{n^3-7n+6} = \lim_{n\to\infty} \frac{\cancel{n^3}}{\cancel{n^3}\left(1-\frac{7}{n^2}+6/n^3\right)}$$

$$= \frac{1}{1} = 1 = L > 0$$

$\sum_{n=2}^{\infty} \frac{1}{n^3-7n+6}$ converges by Limit comparison test.

b) $\displaystyle\sum_{n=2}^{\infty} \underbrace{\frac{1}{(n^4-3)^{1/3}}}_{a_n}$ compare with $\displaystyle\sum_{n=2}^{\infty} \underbrace{\frac{1}{n^{4/3}}}_{b_n}$

converges

p-series
$p = 4/3 > 1$
converges.

$$\lim_{n\to\infty} \frac{n^{4/3}}{(n^4-3)^{1/3}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^{4/3}}}{\left[\cancel{n^4}(1-3/n^4)\right]^{1/3}} = \lim_{n\to\infty} \frac{1}{(1-3/n^4)^{1/3}} = 1$$

$$1 = L > 0$$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+n^{1/3}}{n+3}$ compare with $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum \frac{1}{n^{1/2}}$

diverges.

$p = 1/2 < 1$
diverges.

$$\lim_{n\to\infty} \frac{\dfrac{\sqrt{n}+n^{1/3}}{n+3}}{\dfrac{1}{n^{1/2}}}$$

$$= \lim_{n\to\infty} \frac{n^{1/2}(n^{1/2}+n^{1/3})}{n+3} = \lim_{n\to\infty} \frac{\cancel{n^{1/2}\cdot n^{1/2}}(1+n^{1/3-1/2})}{\cancel{n}(1+3/n)}$$

$$= 1/1 = 1$$

If a_n = ratio of two polynomials

$\sum b_n$ = p-series.

Ex 2 $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + e^n}{3^n + n^2}$ converges. compare $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{e}{3}\right)^n$

geom. $r = e/3 < 1$ converges.

$$\lim_{n\to\infty} \frac{2^n+e^n}{3^n+n^2} \Big/ \frac{e^n}{3^n}$$

$$= \lim_{n\to\infty} \frac{3^n(2^n+e^n)}{e^n(3^n+n^2)} =$$

$$= \lim_{n\to\infty} \frac{\cancel{3^n \cdot e^n}\left(\left(\frac{2}{e}\right)^n + 1\right)}{\cancel{e^n \cdot 3^n}\left(1 + n^2/3^n\right)} = 1.$$

$n^2/3^n \to 0$ L'Hop.

Ex 3. a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 5^n}$ compare to $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{5^n}$

converges.

b) $\sum_{n=2}^{\infty} \frac{\sqrt{n}+7}{n^{3/2}\ln n+16}$ compare to $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^{3/2}\ln n} = \sum_{n=2}^{\infty} \frac{1}{n\ln n}$

$\sum \frac{1}{n(\ln n)^p} = \begin{cases} \text{conv.} & p>1 \\ \text{div} & p\le 1 \end{cases}$

diverges

integral test

c) $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n}$ compare to $\sum_{n=1}^{\infty} \frac{1/n}{n} = \sum \frac{1}{n^2}$

$\lim_{x\to 0} \frac{\sin x}{x} = 1 \Longleftrightarrow x \to 0$ $\sin x \approx x$

conv.

Random series.

a) $\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n}$ limit comparison with $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$.

diverges

b) $\sum_{n=1}^{\infty} \frac{1}{3^n+2} < \sum_{n=1}^{\infty} \frac{1}{3^n}$ conv.

conv.

direct comparison or limit comparison

c) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n \leftarrow$ geom.

conv.

$r = -\frac{2}{3}$

$|r| = \frac{2}{3} < 1$.