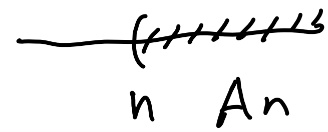
Lecture: 04/01 Lecture

Chapter 17. More on sets (continued)

Example 1. $A_n = \{x \in \mathbb{R} \mid x > n\}, n = 1, 2, \ldots$

a) $\bigcup_{n=1}^{\infty} A_n = (1, +\infty) = A_1$



Proof: • $\bigcup_{n=1}^{\infty} A_n \subseteq (1, +\infty)$

Pick arbitrary $x \in \bigcup_{n=1}^{\infty} A_n \Rightarrow$ there is k, s.t. $x \in A_k \Rightarrow x > k \geq 1 \Rightarrow x \in A_1 = (1, +\infty)$.

• $(1, \infty) \subseteq \bigcup_{n=1}^{\infty} A_n$

Pick arbitrary $x \in (1, \infty)$ $(\forall x \in (1, \infty))$

for every x in $(1, \infty)$

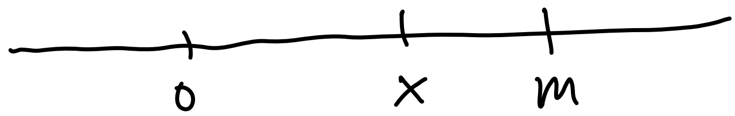
$\Rightarrow x \in A_1 \Rightarrow \bigcup_{n=1}^{\infty} A_n$.

b) $\bigcap_{n=1}^{\infty} A_n = \emptyset$

Need to show that $\bigcap_{n=1}^{\infty} A_n$ has no elements.

By contradiction, assume $x \in \bigcap_{n=1}^{\infty} A_n \Rightarrow$

$x \in A_n$ for all $n = 1, 2, 3, \ldots$



By Arch. principle $\exists m \in \mathbb{N}$, s.t. $x < m$

there is

$$\Rightarrow x \notin A_m \Rightarrow x \notin \bigcap_{n=1}^{\infty} A_n. \text{ (contradiction)}$$

<u>Ex:</u> $A_n = \{x \in \mathbb{R} \mid 1/n < x < \sqrt{3} + 1/n\}$, $n = 1, \ldots$

a) $\bigcup_{n=1}^{\infty} A_n = \underline{(0, \sqrt{3}+1)}$

- $\bigcup_{n=1}^{\infty} A_n \subseteq (0, \sqrt{3}+1)$.

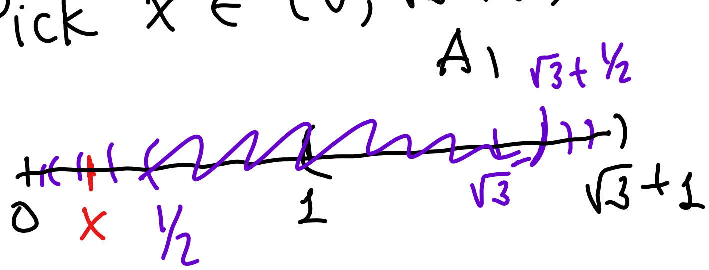
$\forall x \in \bigcup_{n=1}^{\infty} A_n \Rightarrow \exists k$, s.t. $x \in A_k \Rightarrow$

$$0 < \frac{1}{k} < x < \sqrt{3} + \frac{1}{k} \leq \sqrt{3} + 1 \Rightarrow$$

$$0 < x < \sqrt{3} + 1 \Leftrightarrow x \in (0, \sqrt{3}+1).$$

- $(0, \sqrt{3}+1) \subseteq \bigcup_{n=1}^{\infty} A_n$

Pick $x \in (0, \sqrt{3}+1)$



Two cases: 1) $1 < x < \sqrt{3}+1 \Rightarrow A_1$

$\Rightarrow x \in \bigcup_{n=1}^{\infty} A_n.$

2) $0 < x \le 1 \Rightarrow \exists K$ s.t $0<\frac{1}{K} < x \Leftarrow (\frac{1}{x} < K)$

By Arch. principle $\exists K \in \mathbb{N}$ s.t.

$0 < 1/K < x \le 1. \Rightarrow x \in A_K$

$A_K = \{x \in \mathbb{R} \mid \frac{1}{K} < x < \sqrt{3} + 1/K\}.$

$\Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$

b) $\bigcap_{n=1}^{\infty} A_n = (1, \sqrt{3}].$

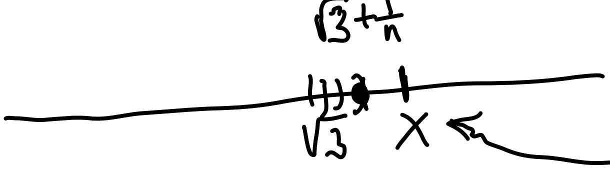
- $\bigcap_{n=1}^{\infty} A_n \subseteq (1, \sqrt{3}]$

Let $x \in \bigcap_{n=1}^{\infty} A_n \Rightarrow x \in A_n$ for all $n = 1, 2, \dots$

$\Rightarrow \boxed{\frac{1}{n} < x < \sqrt{3} + 1/n \text{ for all } n.}$

If $n = 1 \Rightarrow x > 1$

$x < \sqrt{3} + \frac{1}{n}$ for all n.



false

Suppose $x > \sqrt{3}$, choose m by Arch. principle s.t. $1/m < x - \sqrt{3} > 0 \Rightarrow x > \sqrt{3} + 1/m \Rightarrow$

it is impossible since $x < \sqrt{3} + 1/n$ for all n.

$\Rightarrow \boxed{x \le \sqrt{3}}$

$\Rightarrow \quad 1 < x \le \sqrt{3}$.

• $(1, \sqrt{3}] \subseteq \bigcap_{n=1}^{\infty} A_n$.

let $x \in (1, \sqrt{3}]$, then for every n

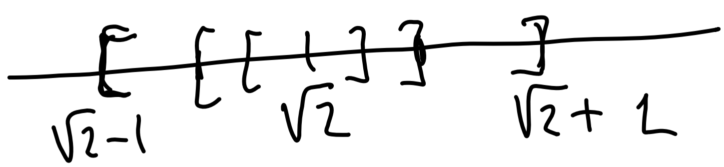
$1/n \le 1 < x \le \sqrt{3} < \sqrt{3} + 1/n$ for all n,

so $x \in A_n$ for all $n \Rightarrow x \in \bigcap_{n=1}^{\infty} A_n$.

($\underline{(1, \sqrt{3}]} \subseteq A_n$ for all n)

<u>Ex:</u> $A_n = \{ \underline{x \in \mathbb{Q}} \mid \sqrt{2} - 1/n \le x \le \sqrt{2} + 1/n \}$

$\bigcup_{n=1}^{\infty} A_n = \mathbb{Q} \cap [\sqrt{2} - 1, \sqrt{2} + 1]$



$$\bigcap_{n=1}^{\infty} A_n = \emptyset = \mathbb{Q} \cap \{\sqrt{2}\}$$