Lecture: Calculus II lecture on 03/31

Quiz #16 (solutions)

a) $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$, $f(x) = \frac{x}{x^4+1} > 0$, continuous and, since

$f'(x) = \frac{x^4+1-4x^4}{(x^4+1)^2} = \frac{1-3x^4}{(x^2+1)^2} < 0$ on $[1,\infty)$, decreasing

$$\int_1^{\infty} \frac{x}{x^4+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^4+1}\,dx \underset{u=x^2}{=} \lim_{t\to\infty} \int_1^{t^2} \frac{\frac{1}{2}du}{u^2+1}$$

$$= \lim_{t\to\infty} \frac{1}{2}\arctan u\Big|_1^{t^2} = \lim_{t\to\infty} \frac{1}{2}\arctan t^2 - \frac{1}{2}\arctan 1 = \pi/4 - \pi/8 = \pi/8$$

So, the series converges.

b) $\sum_{n=1}^{\infty} \frac{1}{2n+1}$, $f(x) = \frac{1}{2x+1} > 0$, cont. and decr. on $[1,+\infty)$

$$\int_1^{\infty} \frac{1}{2x+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{2x+1}\,dx = \lim_{t\to\infty} \frac{1}{2}\ln|2x+1|\Big|_1^t =$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln(2t+1) - \frac{1}{2}\ln 3 = +\infty$$, so the series diverges to $+\infty$.

c) $\sum_{n=1}^{\infty} n e^{-n/4}$, $f(x) = xe^{-x/4} > 0$, cont. Observe that

$f'(x) = e^{-x/4} - \frac{1}{4}xe^{-x/4} = e^{-x/4}(1 - 1/4\,x) \le 0$ when $1 - \frac{1}{4}x \le 0$ or $x \geq 4$, so we can apply the Integral test on $[4,+\infty)$

$$\int_4^{\infty} xe^{-x/4} = \lim_{t\to\infty} \int_4^t xe^{-x/4}\,dx \underset{\text{by parts}}{=} \lim_{t\to\infty} -4xe^{-x/4}\Big|_4^t + \int_4^t 4e^{-x/4}\,dx$$

$$= \lim_{t\to\infty} \left(-4xe^{-x/4} - 16e^{-x/4}\right)\Big|_4^t = \lim_{t\to\infty} -4te^{-t/4} - 16e^{-t/4} + 16e^{-1} + 16e^{-1}$$

$$= \frac{32}{e}$$, series converges.

#48 $\sum_{n=2}^{\infty} \frac{\ln n}{n^p} > 0$, cont, $f'(x) = \left(\frac{\ln x}{x^p}\right)' =$

$= \frac{x^{p-1} - px^{p-1} \cdot \ln x}{x^{2p}}$ $p > 0$

$= \frac{x^{p-1}(1 - p\ln x)}{x^{2p}}$

L'Hop.

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = \begin{cases} 0 & , p > 0 \\ +\infty & , p \leq 0 \end{cases}$$

When $p \leq 0$, series diverges by div. test.

$\ln x > \frac{1}{p}$, $x > e^{1/p}$

$[e^{1/p}, +\infty)$ we can apply integral test

$$\lim_{t \to \infty} \int_N^t \frac{\ln x}{x^p} dx = \lim_{t \to \infty} \int_{\ln N}^{\ln t} u/(e^u)^p \, e^u du$$

$u = \ln x$

$x = e^u$

$du = \frac{1}{x} dx$

$$= \lim_{t \to \infty} \int_{\ln N}^{\ln t} \underbrace{ue^{-up+u}}_{ue^{u(1-p)}} du$$

$$= \begin{cases} +\infty & , \text{when } 1-p \geq 0, \ p \leq 1 \\ \text{number} & , \text{when } 1-p < 0, \ p > 1 \end{cases}$$

#55. $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx \sum_{n=1}^{5} \frac{1}{n^2} + \underbrace{\sum_{n=6}^{\infty} \frac{1}{n^2}}_{\text{error}}$

Integral test to $[6, +\infty)$

$$\int_6^{\infty} \frac{1}{x^2} dx < \underbrace{\sum_{n=6}^{\infty} \frac{1}{n^2}}_{\text{Error}} < \int_6^{\infty} \frac{1}{x^2} dx + \frac{1}{6^2}$$

$$\boxed{\text{Error}} < -\frac{1}{x}\Big|_6^{\infty} + \frac{1}{6^2} = \frac{1}{6} + \frac{1}{6^2} = 7/36.$$

Sec. 9.4 Comparison of series.

Direct comparison test (thm. 9.12):

Let $0 \le a_n \le b_n$ for all n. (or for all $n \geq N$)

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges

Explanation: Since $a_n \geq 0$ and $b_n \geq 0$, for each series there are two possibilities

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \text{some number } A \text{ (converges)}, \\ +\infty \text{ (diverges)}; \end{cases}$$

$$\sum_{n=1}^{\infty} b_n = \begin{cases} B \text{ (converges)}, \\ +\infty \text{ (diverges)} \end{cases}$$

1) If $\sum_{n=1}^{\infty} b_n = B$ (converges) and since $0 \le a_n \le b_n$,

$\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n = B$, so $\sum_{n=1}^{\infty} a_n$ must be finite,

and the series converges

2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} a_n = +\infty$ and

since $0 \le a_n \le b_n$,

$\infty = \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n \Rightarrow \sum_{n=1}^{\infty} b_n = \infty$, so $\sum_{n=1}^{\infty} b_n$ diverges.

Examples: Use Direct Comparison test to decide convergence or divergence.

a) $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+n}}$ compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$n^2+n > n^2$

$\frac{1}{n^2+n} < \frac{1}{n^2}$

Since for all n, $0 \le \frac{1}{n^2+n} < \frac{1}{n^2}$ and

$\sum \frac{1}{n^2+n} < \sum \frac{1}{n^2}$ number

the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p = 2$, $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ also converges

(Note that comparing $\frac{1}{n^2+n} < \frac{1}{n}$ does not work, since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ and all we can conclude from this comparison is that $\sum_{n=1}^{\infty} \frac{1}{n^2+n} \le \infty$.)

$\sum \frac{1}{n^2+n} < \sum \frac{1}{n} = \infty$

b) $\boxed{\sum_{n=1}^{\infty} \frac{1}{4n-1}}$. Compare to $\frac{1}{4n}$

Since $\frac{1}{4n} \le \frac{1}{4n-1}$ for all n,

$\infty = \sum_{n=1}^{\infty} \frac{1}{4n} \le \sum_{n=1}^{\infty} \frac{1}{4n-1}$, $\frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$,

so $\sum_{n=1}^{\infty} \frac{1}{4n} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n} = +\infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n-1}$ diverges

$= \infty$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{3\sin n + 4}{5^n + \sqrt{n}}$ like $\displaystyle\sum_{n=1}^{\infty} \frac{}{5^n}$

Expect to converge

$$\underbrace{\sum_{n=1}^{\infty} \frac{3\sin n + 4}{5^n + \sqrt{n}}}_{\text{converges}} < \underbrace{\sum_{n=1}^{\infty} \frac{7}{5^n}}_{\text{geometric}} = \frac{7/5}{1 - 1/5} < \infty$$

$$\sum_{n=1}^{\infty} \frac{7}{5^n} = 7/5 + 7/25 + \ldots$$

$a = 7/5, \ r = 1/5$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \ln n} \sim \sum \frac{1}{n}$

Expect to diverge

$\boxed{n + \ln n < 2n}$

$$\infty = \sum \frac{1}{2n} < \underbrace{\sum \frac{1}{n + \ln n}}_{\text{diverges.}}$$

e) $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n - 10n + 50}} \sim \displaystyle\sum \frac{1}{e^n}$ geom.

converges

$$\underbrace{\sum \frac{1}{e^n - 10n + \cancel{50}}}_{\text{converges.}} < \sum \frac{1}{\frac{1}{2}e^n + \underbrace{\boxed{\frac{1}{2}e^n - 10n}}_{>0 \ n>5}} < \underbrace{\sum \frac{1}{\frac{1}{2}e^n}}_{\text{conv. geom}}$$

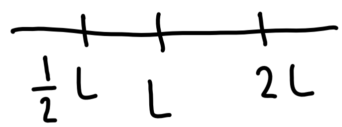
Limit comparison test (theorem 8.13):

Suppose $a_n > 0$ and $b_n > 0$ for all n and $\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right) = L > 0$. Then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

Idea: If $\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right) = L$, then sequences (a_n) and (b_n) have similar behavior as $n \to \infty$.

Explanation: • Observe that convergence or divergence of a series depends only on the behavior of a "tail" of a series

• If $\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right) = L > 0$, $L \neq \infty$, then there is an N such that for all $n \geq N$,



$$\frac{1}{2} L \leq \frac{a_n}{b_n} \leq 2L$$

Then we have
$$0 \leq \frac{1}{2} L \cdot b_n \leq a_n \leq 2L \cdot b_n \quad \text{for all } n \geq N.$$

If $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} \frac{1}{2} L \cdot b_n = \frac{1}{2} L \sum_{n=1}^{\infty} b_n$ also converges by the direct comparison test

So $\sum_{n=N}^{\infty} a_n$ converges $\Rightarrow \sum_{n=N}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} b_n$ converges.

If $\sum_{n=N}^{\infty} a_n = \infty$ (diverges) then the series

$\sum_{n=N}^{\infty} 2L b_n = 2L \sum_{n=N}^{\infty} b_n$ also diverges $\Rightarrow$

$\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges

Similar argument also works to show that

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges and

$\sum_{n=1}^{\infty} b_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

Limit comparison test works especially well in comparing "messy" algebraic series with the standard series.

<u>Q1</u>: What happens if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

<u>Q2</u>: What happens if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$.