Lecture on 03/30

Lecture:

Class announcements.

1. Exam on Friday in class.

2. Last homework was graded and it is in your box

3. Homeworks due today at 3:30 pm and due Wednesday.

Equality in Cauchy inequality:

$\vec{A} = (a_1, \ldots, a_n)$, $\vec{B} = (b_1, \ldots, b_n)$

$0 \leq (\vec{A} - t\vec{B}) \cdot (\vec{A} - t\vec{B}) = \vec{A} \cdot \vec{A} - 2t\vec{A} \cdot \vec{B} + t^2 \vec{B} \cdot \vec{B}$

$= \|A\|^2 - 2\vec{A} \cdot \vec{B} t + t^2 \|B\|^2$

$at^2 + bt + c$

$D = b^2 - 4ac$

$D = 4(\vec{A} \cdot \vec{B})^2 - 4\|\vec{A}\|\|\vec{B}\|^2 \leq 0$

Equality when $D = 0 \iff \|\vec{A} - t\vec{B}\|^2 = 0$

$\Leftrightarrow \boxed{\vec{A} = t\vec{B}}$ for some $t \in \mathbb{R}$.

Note that we proved:

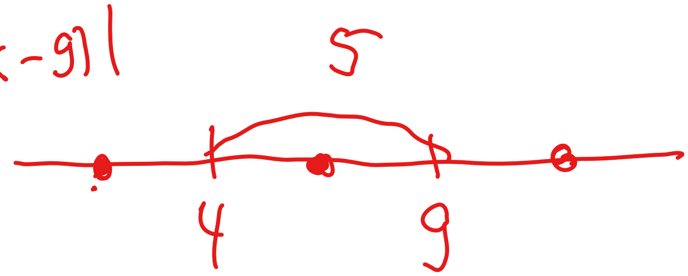
$$|a_1b_1 + a_2b_2 + \dots + a_nb_n| \overset{=}{\leq} \sqrt{a_1^2 + \dots + a_n^2} \cdot \sqrt{b_1^2 + \dots + b_n^2}$$

Homework questions and comments.

Exercise 4d: $|x-4| - |x-9| = 6$

$|a| - |b| \leq |a-b|$ (Corollary 5, part 4)

$|x-4| - |x-9| \leq |x-4-(x-9)|$
$= 5$



So $|x-4| - |x-9| \neq 6$

$\boxed{|x-4| = 6 + |x-9|}$

$|a+b| \leq |a| + |b|$

\# 1. a) $|x-y| \leq \underbrace{|x-z|}_{a} + \underbrace{|z-y|}_{b}$

b) $|(a+b)+c| \leq |a| + |b| + |c|$.

Chapter 17. More on sets (continued)

Definition:
- $A \subseteq B$ (set A is a subset of a set B) if every element x which is in A is also in B.
  ($\forall x$, $x \in A \Rightarrow x \in B$).
- $A \subset B$ (set A is proper subset of a set B) if $A \subseteq B$, $A \neq B$.
- $A = B$ (two sets A and B are equal) if $A \subseteq B$ and $B \subseteq A$.

Checking that two sets are equal

Step 1. Check that if $x \in A$, then $x \in B$, for all x

Step 2. Check that if $y \in B$, then $y \in A$, for all y.

Proposition 17.1 Let A, B, C be any sets then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: **Step 1.** Let $x \in A \cap (B \cup C)$

$\Leftrightarrow$ $x \in A$ and $x \in (B \cup C)$ $\Leftrightarrow$

$\Leftrightarrow$ $x \in A$ and $(x \in B$ or $x \in C)$

$\Leftrightarrow$ $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow$ $(x \in A \cup B)$ and $(x \in A \cup C)$

$\Leftrightarrow$ $x \in (A \cup B) \cap (A \cup C)$

Step 2.

Notation:

- Union of sets $A_1, A_2, \dots, A_n$ is

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \dots \cup A_n$$
$$= \{x \mid x \in A_i, \text{ for some } i\}$$

- Intersection of sets $A_1, A_2, \dots, A_n$ is $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \dots \cap A_n$

$$= \{x \mid x \in A_i, \text{ for all } i = 1, \dots, n\}$$

<u>Ex</u>: Find sets A, B, C such that

$A \cap B = \emptyset$, $A \cap C \neq 0$, $B \cap C \neq 0$

(so A, B, C are disjoint) and

$A \cup B \cup C = \mathbb{N}$, <u>$A, B, C$ are infinite.</u>

$A = \{2, 4, 6, \dots\} \leftarrow$ evens

$B = \{3, 5, 7, 11, \dots\} \leftarrow$ odd primes

$C = \{1, 9, 15, \dots\}$

- Now find 20 sets like this.

$A_0, A_1, \dots, A_{19}$, $0, 1, \dots, 19$ are the remainder

$A_i = \{20k + i \mid k \in \mathbb{N}\}$, $A_1 = \{1, 21, 41, \dots\}$

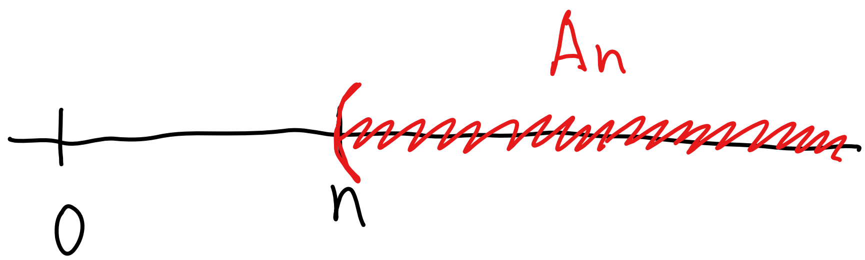
<u>Notation</u>: $\bigsqcup_{i=1}^{n} A_i$ ← disjoint union of sets $A_1, \dots, A_n$.

$\parallel$

$\bigcup_{i=1}^{\infty} A_i$, $A_i \cap A_j = \emptyset$ for any i and j

<u>Ex</u>: Work out $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$.

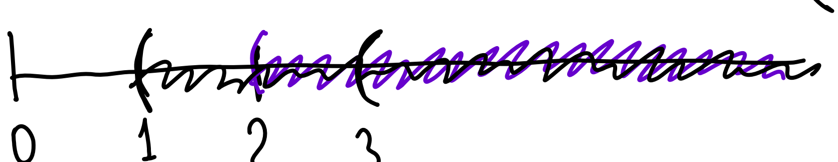
a) $A_n = \{x \in \mathbb{R} \mid x > n\}$



- $\bigcup_{n=1}^{\infty} A_n = (1, +\infty) = A_1$

$A_1 \supset A_2 \supset A_3 \supset \dots$

↖ ↑ ↗ nested sets.

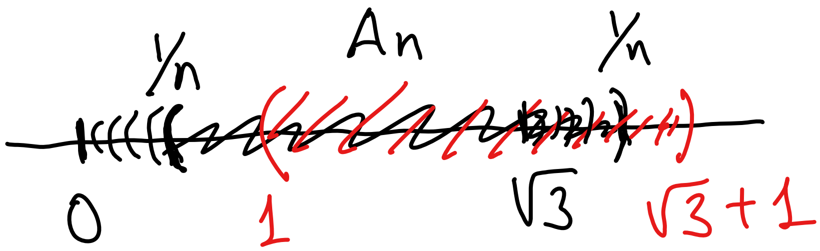


$A_1 \cup A_2 = A_2$, $A_2 \subset A_1$

- $\bigcap_{n=1}^{\infty} A_n = \emptyset$

$$1000 \notin \bigcap_{n=1}^{\infty} A_n \text{ , because } 1000 \notin A_{1001}$$

b) $A_n = \{ x \in \mathbb{R} \mid \frac{1}{n} < x < \sqrt{3} + 1/n \}$



- $\bigcup_{n=1}^{\infty} A_n = (0, \sqrt{3}+1)$

- $\bigcap_{n=1}^{\infty} A_n = (1, \sqrt{3}]$