Lecture: Calculus lecture on 03/30.

Class announcements:

1. Please try to attend lectures.
2. Homework 16 15 graded and can be found in the folder Calc2_YourLastName. Please let me know if you submitted homework or quiz, but it is missing from that folder.
3. Exam 2 will be on Thursday during class.
4. There will be a quiz today at 1.35 p.m.
5. Hwk 19 will be due Tuesday and there will be a quiz on this hwk also on Tuesday

Questions about hwk 18.

#22 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+2n^2+1} = f(n)$

$$f(x) = \frac{x}{x^4+2x^2+1} > 0, \text{ continuous}$$

$f(x)$ is decreasing, $f'(x) < 0$ on $[1, +\infty)$

$$f(x) = \frac{1}{x^3+2x+\frac{1}{x}} \quad g(x)$$

$$g'(x) = 3x^2+2-\frac{1}{x^2} > 0 \quad x > 1.$$

We can apply int. test.

$$\int_1^{\infty} \frac{x}{x^4+2x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{\underbrace{x^4+2x^2+1}_{(x^2+1)^2}}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$$= \lim_{t\to\infty} \int_2^{t^2+1} \frac{\frac{1}{2}\,du}{u^2} < \infty.$$

Series converges.

$\boxed{\#10}$ $f(x) = \dfrac{\ln x}{x} > 0$, cont. $f'(x) = \dfrac{\frac{1}{x}\cdot x - \ln x}{x^2} = \dfrac{1-\ln x}{x^2} < 0$

$e^{\ln x} > e^1$, $\boxed{x > e}$ One can apply ~~int. test~~ $\left(\sum_{n=3}^{\infty} \frac{\ln n}{\sqrt{n}}\right)$ to $\int_3^{\infty} \frac{\ln x}{x}\,dx = +\infty$ diverges

Sec. 9.3 (handout) and 5.3 (book)
Integral Test and p-series (continued)

Recall

The integral test: If f is continuous, positive, and decreasing function where $a_n = f(n)$ on the interval $[1, +\infty)$, then the improper integral $\int_1^\infty f(x)\,dx$ and the infinite series $\sum_{n=1}^\infty a_n$ either both converge or both diverge to $+\infty$

In the case of convergence,

$$\int_1^\infty f(x)\,dx < \sum_{n=1}^\infty a_n < \int_1^\infty f(x)\,dx + a_1 .$$

We will apply this test to a p-series:

$$\boxed{\sum_{n=1}^\infty \frac{1}{n^p}, \quad p > 0.} \qquad \frac{1}{n^p} = f(n)$$

The function $f(x) = \frac{1}{x^p}$ is positive, continuous, and decreasing on $[1, +\infty)$ and the improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & p > 1 \\ +\infty, & 0 < p \le 1 \end{cases}$$

(see textbook or class notes for sec. 3.7. for the derivation of this integral).

When $p \leq 0$, $\frac{1}{n^p} = n^{-p} \geq 1$, so $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$
$-p \geq 0$
and the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the divergence test.

To summarize:

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges to $+\infty$ when $p \leq 1$.

- If $p > 1$, then
$$\frac{1}{p-1} < \sum_{n=1}^{\infty} \frac{1}{n^p} < \frac{1}{p-1} + 1 = \frac{p}{p-1}.$$

Examples: • $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, converges by p-series test.
Moreover, $1 = \frac{1}{2-1} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{2}{2-1} = 2$.

- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} \leq 1$, so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges to $+\infty$.

- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ is not a p-series.

• $\sum_{n=1}^{\infty} \sqrt{n}$ diverges by the divergence test, since $\lim_{n\to\infty} \sqrt{n} \neq 0$.

$\sqrt{n} = n^{1/2} = \frac{1}{n^{-1/2}}$, $p = -1/2 < 0$.

Additional examples:

a) $\sum_{n=1}^{\infty} (-3/2)^n$ geom. $a = -3/2$, $r = -3/2$

$= -3/2 + (-3/2)^2 + \ldots = a + ar + \ldots$

$|r| = |-3/2| = 3/2 > 1$ series diverges.

b) $\sum_{n=1}^{\infty} (-2/3)^n$ geom. $a = -2/3$, $r = -2/3$

converges to $\frac{a}{1-r} = \frac{-2/3}{1+2/3} = \frac{-2}{5}$

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ ← diverges.

Integral test $\int_1^{\infty} \frac{1}{\sqrt{x+1}}\,dx$

$= \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x+1}}\,dx$, $\frac{1}{\sqrt{x+1}} = (x+1)^{-1/2}$

$= \lim_{t\to\infty} \left. \frac{(x+1)^{1/2}}{1/2} \right|_1^t =$

$$= \lim_{t \to \infty} 2(t+1)^{1/2} - 2 \cdot 2^{1/2} = +\infty$$

d) $\displaystyle\sum_{n=1}^{\infty} n e^{-n/10}$ ← converges.

Integral test:

$f(x) = x e^{-x/10} > 0$, cont.

$$f'(x) = e^{-x/10} - \frac{1}{10} x e^{-x/10}$$

$$= \underbrace{e^{-x/10}} \left(1 - \frac{1}{10} x\right) < 0 \quad 0$$

$1 - \frac{1}{10} x < 0$, $\quad x > 10$

$$\sum_{n=10}^{\infty} n e^{-n/10}$$

$$\int_{10}^{\infty} x e^{-x/10} \, dx < \infty$$