Calculus lecture on 03/27

Lecture:

Class announcements.

- Quiz 15 is graded and it is in your box.
- Exam 2 will be on Thursday during class. Same format as quiz 15.
- Homework 18 is due Monday. There will be a quiz on this hwk also on Monday.
- Will post links to lectures 03/26 and 03/27 and to two videos on integral test in homework 18 assignment.

Solution to quiz 15

a) $\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n = 2/5 + \left(\frac{2}{5}\right)^2 + \ldots = a + ar + \ldots$

Geometric series with $a = 2/5$ and $r = 2/5$

Converges to $\frac{a}{1-r} = \frac{2/5}{1-2/5} = \frac{2}{3}$.

$$\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n = 2/3.$$

b) $\sum_{n=1}^{\infty} \frac{n}{n+1000}$. $a_n = \frac{n}{n+1000}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1000} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(1+\frac{1000}{n}\right)}$$

$$= 1 \neq 0$$

Series diverges by the divergence test.

c) $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

Since $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$,

the series is telescoping.

$$S_K = \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \cdots + \frac{1}{K(K-1)} =$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \cancel{\left(\frac{1}{2} - \frac{1}{3}\right)} + \cdots + \cancel{\left(\frac{1}{K-1} - \frac{1}{K}\right)}$$

$$= 1 - 1/K.$$

$\lim_{K \to \infty} S_K = 1 \Rightarrow$ series converges to 1.

Sec 9.3 (handout) and 5.3 (book).

Integral test and p-series.

Integral test: If f is continuous, positive, and decreasing function where $a_n = f(n)$ on the interval $[1, +\infty)$, then the improper integral $\int_1^\infty f(x)\,dx$ and the infinite series $\sum_{n=1}^\infty a_n$ either both converge or both diverge to $+\infty$.

Also: when the integral and the series both converge, then

$$\int_1^\infty f(x)\,dx < \sum_{n=1}^\infty a_n < \int_1^\infty f(x)\,dx + a_1$$

Example: Use integral test to decide convergence or divergence of each series. If series converges, estimate its sum.

a) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+1}$, $a_n = f(n)$, $f(x) = \frac{1}{x^2+1}$.

$a_n = \frac{1}{n^2+1}$

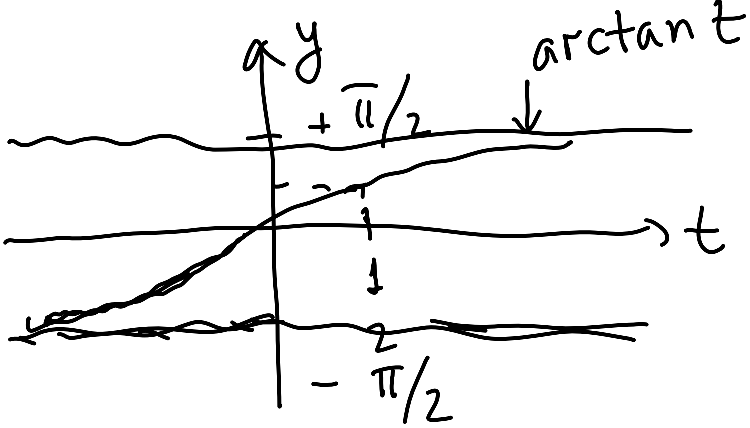
$f(x) = \frac{1}{x^2+1} > 1$, decreasing on $[1, +\infty)$, also continuous

$$\underbrace{\int_1^{+\infty} \frac{1}{x^2+1}\,dx}_{\text{improper}} = \lim_{t\to+\infty} \int_1^t \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \arctan x \Big|_1^t$$

$$= \lim_{t\to\infty} \arctan t - \arctan 1$$

$$= \pi/2 - \pi/4 = \pi/4 < \infty$$



By the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

and

$$\pi/4 < \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2+1}}_{} < \frac{\pi}{4} + \frac{1}{1^2+1} = \pi/4 + \frac{1}{2}$$

b) $\sum_{n=1}^{\infty} \frac{1}{n}$, $f(x) = \frac{1}{x} > 0$, decreasing, continuous on $[1, +\infty)$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x}\,dx$$

$$= \lim_{t \to \infty} \ln|x| \Big|_1^t = \lim_{t \to \infty} \ln|t| - \ln 1$$

$$= +\infty$$

By integral test, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to $+\infty$

c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, $f(x) = \frac{1}{x(\ln x)^2} > 0$, decreasing and continuous on $[2, +\infty)$

Applying the integral test on $[2, +\infty)$ not $[1, +\infty)$!

$$\int_2^{+\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2}\,dx$$

$$\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{du}{u^2} = -u^{-1} + C = -\frac{1}{\ln x} + C$$

$u = \ln x,\ du = \frac{1}{x}dx$

$$= \lim_{t\to\infty} -\frac{1}{\ln x}\Big|_2^t$$

$$= \lim_{t\to\infty} -\cancel{\frac{1}{\ln t}}^{\,0} - \left(-\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$

By the integral test series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges

$$\frac{1}{\ln 2} < \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} < \frac{1}{\ln 2} + \frac{1}{2(\ln 2)^2}$$

d) $\sum_{n=1}^{\infty} ne^{-n}$, $f(x) = xe^{-x} > 0$, continuous

Decreasing on $[1,\infty)$?

$f'(x) = e^{-x} - xe^{-x} = \underbrace{e^{-x}}_{>0}\underbrace{(1-x)}_{\leq 0} \leq 0$

$f(x)$ is decreasing.

$\int_1^{\infty} xe^{-x}\,dx$ = compute yourself

$\int_1^{\infty} xe^{-x}\,dx < \infty$, series converges

$$\int_1^{\infty} xe^{-x}dx < \sum_{n=1}^{\infty} ne^{-n} < \int_1^{\infty} xe^{-x}dx + e^{-1}$$

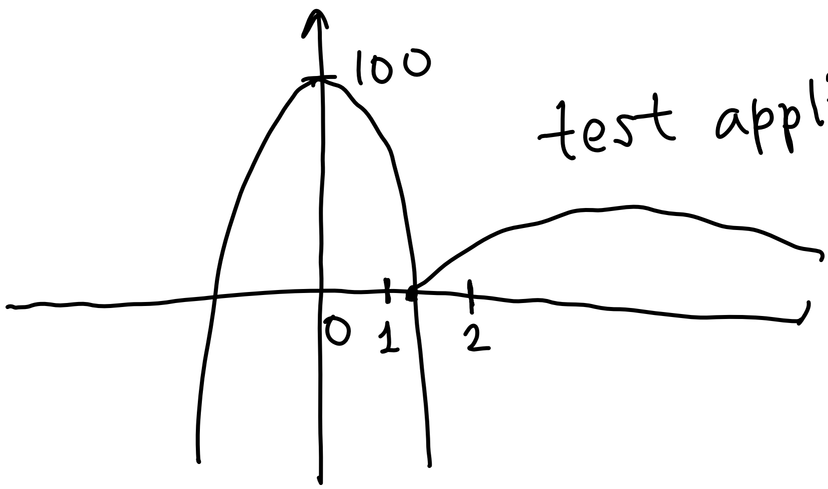
e) $\sum_{n=1}^{\infty} (n^2+100n)e^{-n}$

$f(x) = (x^2+100x)e^{-x} > 0$, continuous

Decreasing: $f'(x) = (2x+100)e^{-x} - (x^2+100x)e^{-x}$

$= \underset{>0}{e^{-x}}\underbrace{(-x^2-98x+100)} < 0$ $\boxed{n \geq 2}$

want this.



test applicable here!

$$\int_2^{\infty} (x^2+100x)e^{-x}\,dx < \infty$$ (find it by integr. by parts)

$$\sum_{n=1}^{\infty} (n^2+100n)e^{-n}$$ converges.

I can estimate

$$\int_2^{\infty} \ldots < \sum_{n=2}^{\infty} (n^2+100n)e^{-n} < \int_2^{\infty} + (2^2+100\cdot 2)e^{-2}$$