UNORDERED SUMMATION

LOREN SPICE

This material is probably in just about any point-set topology book, but I learned it out of Kelley's *General Topology*. Nets are defined and discussed in Chapter 2, and unordered sums are analysed in Problem #2G.

For Homework #3.1, we need to be able to handle summations over an *uncountable* index set. To make sense of this, recall that *all* infinite summations are defined by limiting processes; we just need to figure out an appropriate limit. The trick is that *sequences* (which are inherently countable) aren't enough for uncountable sums; we need to consider *nets*. These objects have a reputation for being mysterious, but they aren't really; we will demonstrate this by eventually translating our net-based definition into concrete, and familiar, terms.

Recall that, for any sequence $(a_i)_{i=1}^\infty$ in $\mathbb{R}$, we have by definition that

$$\sum_{i=1}^\infty a_i = \lim_{N\to\infty} \sum_{i=1}^N a_i.$$

In more concrete terms, we are approximating our *infinite* sum by ever-larger *finite* sums; and this, it turns out, works just fine as a definition in general. That is, for *any* indexing set I and any sequence $(a_i)_{i\in I}$ in $\mathbb{R}$, we put

$$\sum_{i\in I} a_i = \lim_{\substack{F\subseteq I \\ F \text{ finite}}} \sum_{i\in F} a_i$$

(if the limit exists), and call $\sum_{i\in I} a_i$ the *unordered sum* of $(a_i)_{i\in I}$. Two natural questions are "What does that funny limit mean?" and "Why 'unordered'?"; we'll address the second first. Note that our definition doesn't even involve an order on I, hence can't possibly be considered ordered; for example, our definition regards the finite set of prime numbers up to 2^{10} as an indexing set for a sum that is just as natural as $\{1, \dots, 172\}$. More formally, it is clear that, if π is any permutation of I, then the unordered sum $\sum_{i\in I} a_i$ exists if and only if the unordered sum $\sum_{i\in I} a_{\pi(i)}$ does, in which case they're equal; there is no interesting theory of *rearrangements* for unordered sums.

Now to "What does that funny limit mean?" To give you an idea that this isn't the first funny limit that you've seen, think back to the definition of the Riemann integral. You probably first saw that defined as some sort of limit of Riemann sums corresponding to partitions whose mesh goes to 0; but what does it mean to take a limit over partitions? Well, that's also a limit over a net.

So, what is a net? It's exactly like a sequence, just with a different indexing set. That is, a net can have *any* indexing set D; but we need to have a notion of 'later terms' in the net, so D needs to carry at least a partial order ($\prec$), and we don't want to be able to wander off in too many different directions in D, so we'd like to know that, for any two terms in the net, there is another that comes later. In terms of the indexing set, this means that we require that, given any $m, n \in D$, there is some $d \in D$ such that $m \prec d$ and $m \prec n$.

For unordered sums, the indexing set for our net is the collection D of finite subsets of I, and the ordering is set containment: $F \prec F'$ if and only if $F \subseteq F'$. Notice that D is certainly not *linearly* ordered (unless I has at most 1 element); but, if $F, F' \in D$, then $F \cup F' \in D$ as well, and $F \prec F \cup F'$ and $F' \prec F \cup F'$.

For Riemann integration on $[a, b]$, the indexing set for our net is the collection D of partitions of $[a, b]$, and the ordering is refinement: $\mathcal{P} \prec \mathcal{Q}$ if $\mathcal{P}$ is a refinement of $\mathcal{Q}$. If we view partitions as just finite subsets, this is again set containment; $\mathcal{P}$ is a refinement of $\mathcal{Q}$ if and only if $\mathcal{P} \subseteq \mathcal{Q}$. It is a commonplace of any rigorous discussion of Riemann integration that any two partitions have a common refinement. Again, if we view partitions as sets, we see that $\mathcal{P} \cup \mathcal{Q}$ is a common refinement of $\mathcal{P}$ and $\mathcal{Q}$.

So far we just have a fancy notion of partial ordering; what shall we do with it? Well, if $(x_i)_{i \in D}$ is a net in $\mathbb{R}$, and $x \in \mathbb{R}$, then the condition for x to be the limit of $(x_i)_{i \in D}$ is a *exactly* the familiar one, with $\mathbb{Z}_{>0}$ changed to D and $(\le)$ changed to $(\prec)$: We say that $\lim_{i \to D} x_i = x$ if and only if, for all $\varepsilon > 0$, there exists $i \in D$ such that, for all $i \prec j$, we have $|x_j - x| < \varepsilon$. We can similarly define what it means for a net to converge to $\pm\infty$.

Limits of nets behave in the ways that you'd expect; a net can have no limits, or exactly one limit, but no more. The definition of a *subnet* is only a little more complicated than that of a subsequence, but the conclusion is the same; a subnet of a convergent net is convergent, with the same limit. A Cauchy net is convergent. Any bounded net has a Cauchy subnet.

So, what does it mean to compute an unordered sum? As we mentioned, an unordered sum over I is just the limit (if it exists) of a net indexed by finite subsets of I. Let's recall the definition. For any collection $(a_i)_{i \in I}$ of real numbers, indexed by I, and any finite subset F of I, let's put $A_F = \sum_{i \in F} a_i$. Then

$$\sum_{i \in I} a_i := \lim_{\substack{F \subseteq I \\ F \text{ finite}}} \sum_{i \in F} a_i = \lim_{\substack{F \subseteq I \\ F \text{ finite}}} A_F,$$

if the limit exists. A direct translation of the definition of convergence gives, for $S \in \mathbb{R}$, that $S = \sum_{i \in I} a_i$ if and only if, for each $\varepsilon > 0$, we may find a finite subset $f \subseteq I$ such that, for all finite sets $F \supseteq f$, we have that $|S - A_F| < \varepsilon$. The Cauchy criterion says that it is enough to check that, for all $\varepsilon > 0$, we can find a finite subset $f \subseteq I$ such that, for all finite sets $F, F' \supseteq f$, we have $|A_F - A_{F'}| < \varepsilon$; *i.e.*,

$$\Big|\sum_{i \in F} a_i - \sum_{i \in F'} a_i\Big| < \varepsilon.$$

One can also define what it means for an unordered sum to be $\pm\infty$.

The unordered sum over a *finite* indexing set is just the ordinary sum. The unordered sum over a *countable* indexing set is the usual sum of a series, if that series is absolutely convergent (*i.e.*, all rearrangements also converge); and otherwise it does not exist. The interesting thing is that these are the *only* examples; you'll be proving a special case of that on Homework #3.1. However, don't worry too much about approaching that problem on the level of definitions; this formal definition is here in the background if you need it, but you should be able to get by with a pretty common-sense approach when you use the hint.