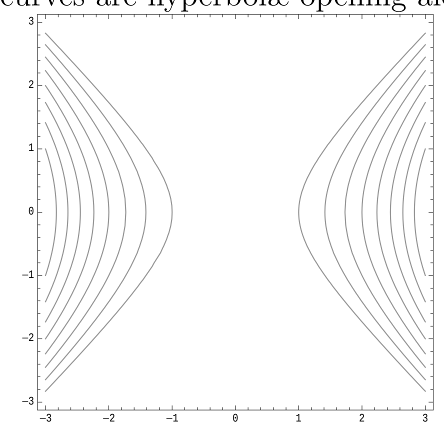
CONTOUR MAPS

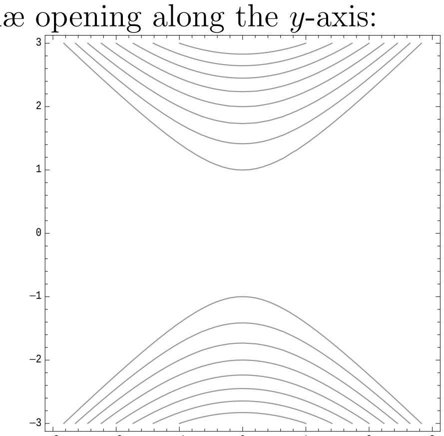
LOREN SPICE

In class, we discussed the fact that a contour map for a function $z = f(x, y)$ of two variables is a picture of what you would see if you looked down on the graph from very high up on the z-axis. We assume that you're so high up that you can't tell the actual height (*i.e.*, the actual z-values) of any of the points, but that you *can* tell when two points have the *same* height. The collection of points at the same height will form a curve (or, in some cases, collection of curves); a cartographer would call these contours, but we call them *level curves*. To find a level curve, you just choose a height $z = c$ and then write down the equation $f(x, y) = c$, where $f(x, y)$ is the formula for the original function.

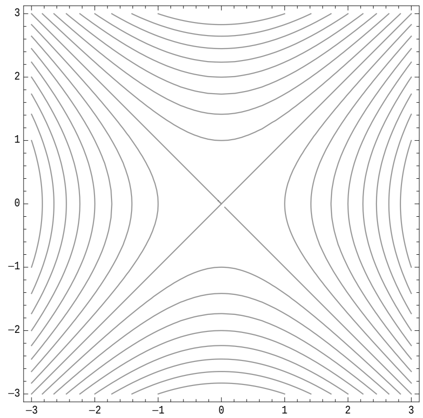
For example, for the function $z = x^2 - y^2$, level curves are the graphs of $x^2 - y^2 = c$, for various values of c. If $c > 0$, then these curves are hyperbolæ opening along the x-axis:



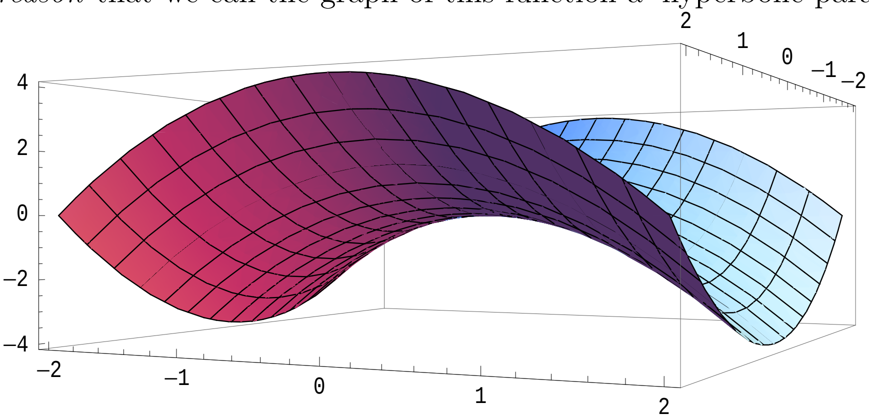
if $c < 0$, then they are hyperbolæ opening along the y-axis:



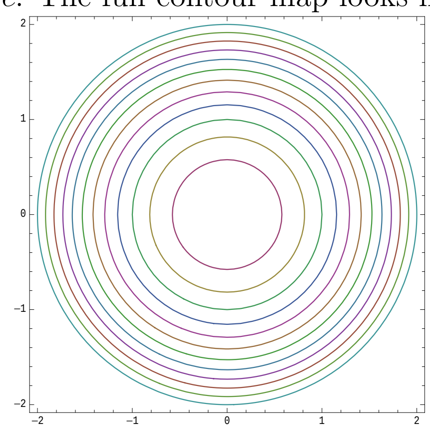
and, if $c = 0$, then we get $x^2 - y^2 = 0$, which gives the two lines $x = 0$ and $y = 0$ (both part of a *single* contour). The full contour map looks like:



This map is the *reason* that we call the graph of this function a 'hyperbolic paraboloid'.



For the function $z = x^2 + y^2$, level curves are the graphs of $x^2 + y^2 = c$, for various values of c. Note that there is no graph at all for $c < 0$; the graph is entirely on or above the xy-plane. For $c = 0$, we get the graph of $x^2 + y^2 = 0$, which is just a point (the origin). For $c > 0$, we get a circle centred at the origin of radius $\sqrt{c}$. The full contour map looks like:

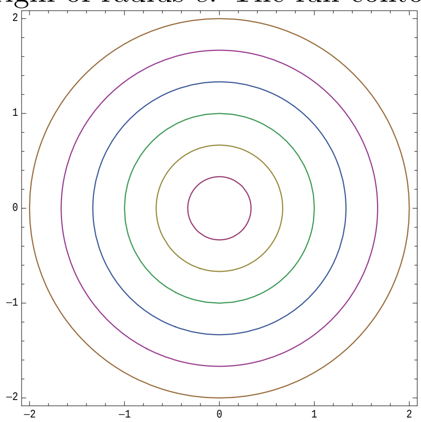


Again, this map is the reason that we call the graph of this function a 'circular paraboloid'. It will be important later to know that we have drawn the contours for $z = 1/3, 2/3, 1, 4/3, \ldots, 10/3, 11/3, 4$. (The contour for $z = 0$ is invisible, but, again, it's just a point at the origin.)

In class, though, we also considered the function $z = \sqrt{x^2 + y^2}$. Again, since this graph is entirely on or above the xy-plane, there's nothing on the level curve $z = c$ unless $c \geq 0$; and, again, the level curve $z = 0$ is just a point. For $c > 0$, we get

$$\sqrt{x^2 + y^2} = c \Leftrightarrow x^2 + y^2 = c^2,$$

which is a circle centred at the origin of radius c. The full contour map looks like:

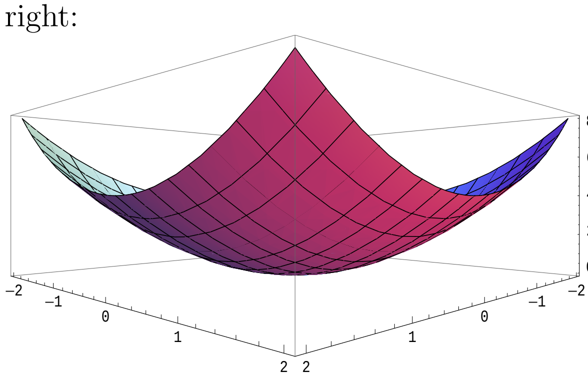


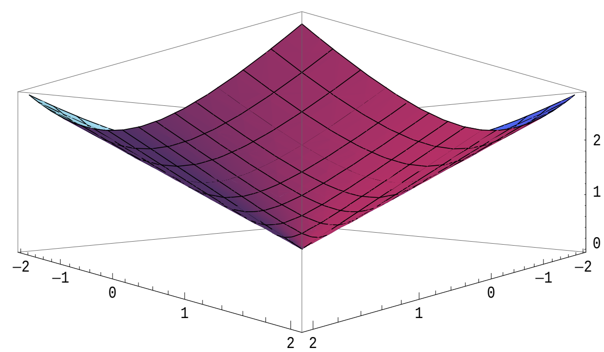
This map is the reason that we call the graph of this function a 'circular cone'. The level curves here are the same as in the previous example, except that we 'run out' sooner; in the same region, we can only fit the level curves for $z = 1/3, 2/3, 1, 4/3, 5/3, 2$.

Notice the difference: the level curves for $z = x^2 + y^2$ bunch up as we move away from the origin, *i.e.*, to bigger z-values, *i.e.*, 'higher' on the graph; whereas the level curves for $z = \sqrt{x^2 + y^2}$ remain equally spaced as we move higher. What this means is that, as we move to the boundaries of the square in the figure (towards $x = \pm 2$ or $y = \pm 2$, we have reached $z = 4$ on a circular paraboloid, but only $z = 2$ on a circular cone. The lesson that we take from this is:

$\boxed{\text{The more 'tightly bunched' the contours, the more quickly the values of the function are changing.}}$

The actual graphs are below, with the circular paraboloid on the left and the circular cone on the right:





In our picture, the tightly bunched contours indicate that the circular paraboloid is *increasing* quickly; but we'd get the same kind of behaviour if it were *decreasing* quickly, so that all we can say for *sure* from looking at the bare contour map is that there is rapid change in *some* direction. (This is why a 'real' contour map will usually be labelled with the actual z-values, not just the curves.)

For future use, notice also that the contours seem to be 'zooming in' on the origin. In terms of calculus and these particular functions, what's so special about the origin? This observation will be useful when we discuss multi-variable optimisation in §13.8.