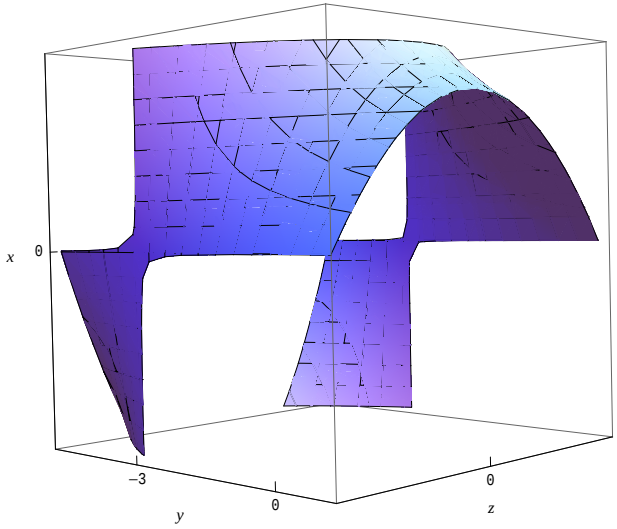
Find the point on the surface

$$xy + 3x + z^2 = 9$$

closest to the origin.



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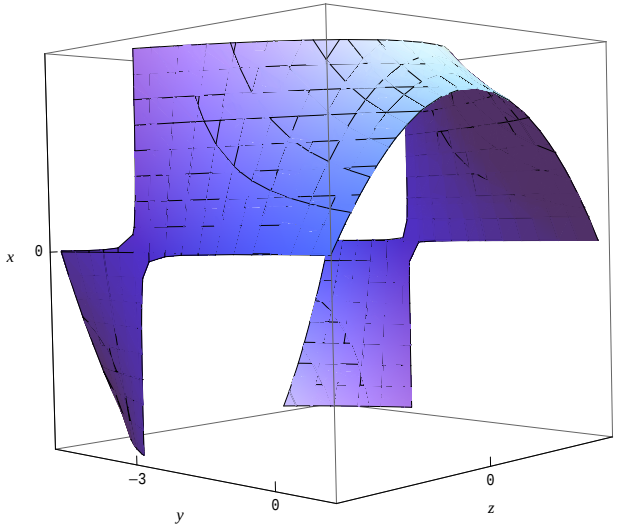

Minimise distance.

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Boundary is where one or more inequalities become equalities:

$$xy + 3x = 9.$$

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Must deal with the *boundary*.

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Find critical points for this one-variable problem:

$$\frac{\mathrm{d}D}{\mathrm{d}y} = -\frac{2 \cdot 9^2}{(y+3)^3} + 2y \stackrel{?}{=} 0.$$

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Ugh!

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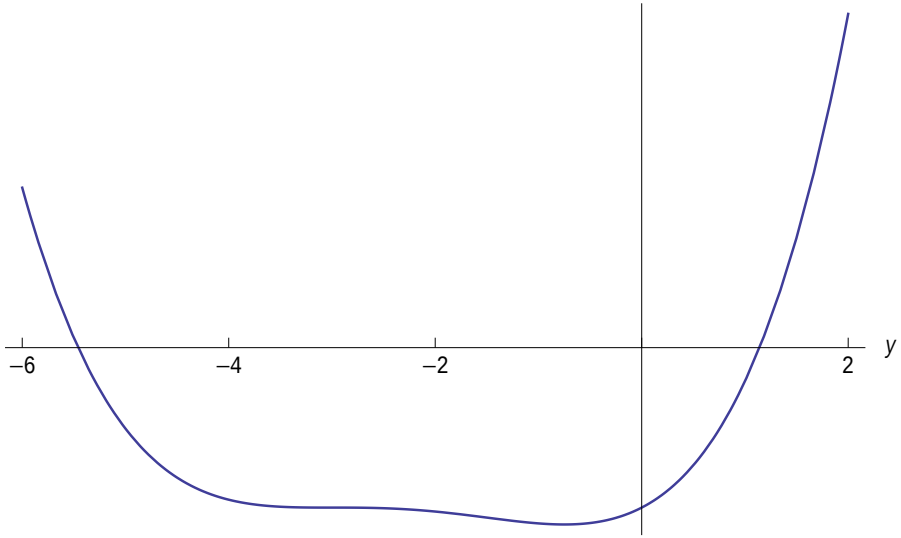

By CAS: $y \approx -5.458, 1.141.$

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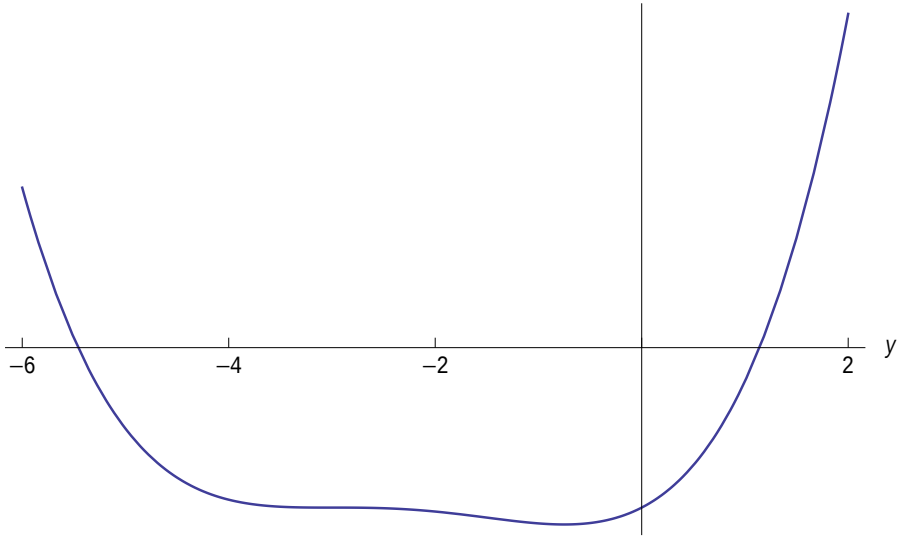

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Always looking for (x, y) pairs: $(x, y) \approx (-3.662, -5.458)$,

$(x, y) = (1.141, 2.173)$.

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Now, take *all* critical points and plug in to

$$D = x^2 + y^2 + (9 - xy - 3x)$$

to find minimum distance ⋆⋆:

x	y	D
2	1	6
1.141	2.173	6.026

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Looking for a point (x, y, z). Since $xy + 3x + z^2 = 9$, get

$$z = \pm\sqrt{9 - xy - 3x} = \pm\sqrt{9 - (2)(1) - 3(2)} = \pm 1.$$

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Closest points are $(2, 1, \pm 1)$, and minimum distance is $\sqrt{D} = \sqrt{6}$.