ITERATED INTEGRALS
CALCULUS III
SPRING 2013

In class, we computed the integral

$$\int_2^3 \int_2^{x+1} (2x^2(y-1)^{-2} + 2y)\mathrm{d}y\mathrm{d}x$$

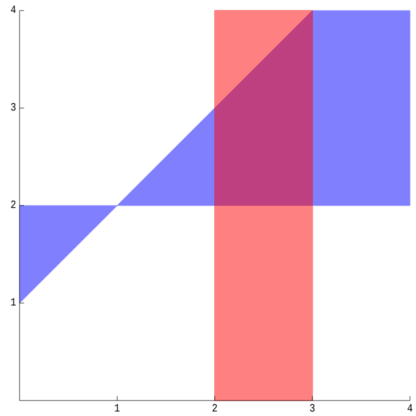
by first computing the inner integral:

$$\begin{aligned}
&\int_2^{x+1} (2x^2(y-1)^{-2} + 2y)\mathrm{d}y \\
&\qquad = (-2x^2(y-1)^{-1} + y^2)\big|_{y=2}^{y=x+1} \\
&\qquad = \left(-2x^2((x+1)-1)^{-1} + (x+1)^2\right) - \left(-2x^2((2)-1)^{-1} + (2)^2\right) \\
&\qquad = \left(-2x^2x^{-1} + (x^2+2x+1)\right) - \left(-2x^2 1^{-1} + 4\right) \\
&\qquad = 3x^2 - 3,
\end{aligned}$$

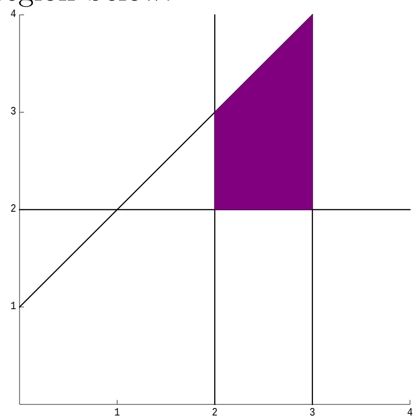
and then plugging it into the outer integral:

$$\begin{aligned}
&\int_2^3 \int_2^{x+1} (2x^2(y-1)^{-2} + 2y)\mathrm{d}y\mathrm{d}x \\
&\qquad = \int_2^3 (3x^2 - 3)\mathrm{d}x \\
&\qquad = (x^3 - 3x)\big|_{x=2}^{x=3} \\
&\qquad = \left((3)^3 - 3(3)\right) - \left((2)^3 - 3(2)\right) \\
&\qquad = (27-9) - (8-6) = 16.
\end{aligned}$$

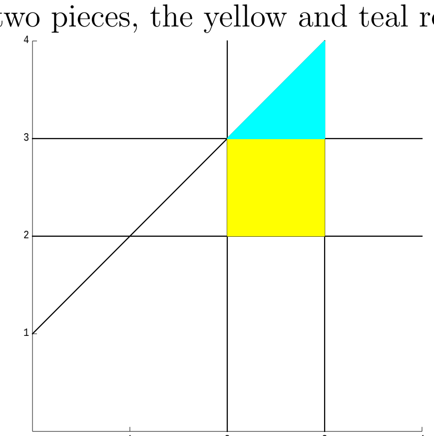
We interpret this as an integral over the region that is *both* between $y = 2$ and $y = x + 1$ (the pale blue region below) *and* between $x = 2$ and $x = 3$ (the pale red region below),



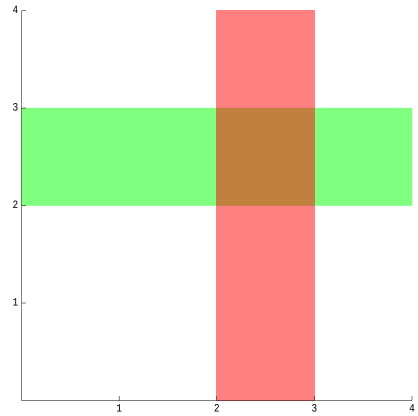
that is, the purple region below:



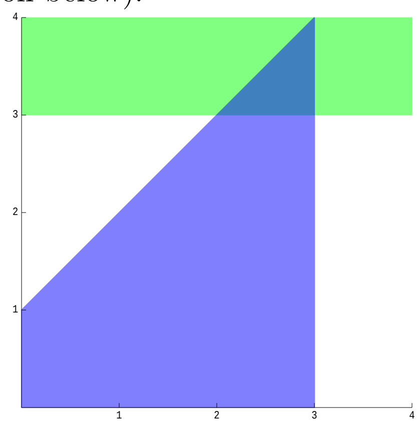
In order to switch the order of integration, we chopped the purple region above into two pieces, the yellow and teal regions below:



We view the yellow region as the region that is *both* between $x = 2$ and $x = 3$ (the pale red region below) *and* between $y = 2$ and $y = 3$ (the pale green region below):



and we view the teal region as the region that is *both* between $x = y-1$ and $x = 3$ (the pale blue region below) *and* between $y = 3$ and $y = 4$ (the pale green region below):



Since we had to chop up our region, we will have to chop up the integral, too, when we switch the order of integration:

$$\int_2^3 \int_2^3 (2x^2(y-1)^{-2} + 2y) \mathrm{d}x \, \mathrm{d}y + \int_3^4 \int_{y-1}^3 (2x^2(y-1)^{-2} + 2y) \mathrm{d}x \, \mathrm{d}y.$$

Note that the variable *upper* limit on the original y integral has become a variable *lower* limit on the new x integral; this kind of change is typical.

To evaluate the switched integral, we first compute the inner integrals:

$$\begin{aligned}\int_2^3 &(2x^2(y-1)^{-2}+2y)\mathrm{d}x\\ &= \left(\frac{2}{3}x^3(y-1)^{-2}+2y\cdot x\right)\Big|_{x=2}^{x=3}\\ &= \left(\frac{2}{3}(3)^3(y-1)^{-2}+2y(3)\right)-\left(\frac{2}{3}(2)^3(y-1)^{-2}+2y(2)\right)\\ &= \left(18(y-1)^{-2}+6y\right)-\left(\frac{16}{3}(y-1)^{-2}+4y\right)\\ &= \frac{38}{3}(y-1)^{-2}+2y\end{aligned}$$

and

$$\begin{aligned}\int_{y-1}^3 &(2x^2(y-1)^{-2}+2y)\mathrm{d}x\\ &= \left(\frac{2}{3}x^3(y-1)^{-2}+2y\cdot x\right)\Big|_{x=y-1}^{x=3}\\ &= \left(\frac{2}{3}(3)^3(y-1)^{-2}+2y(3)\right)-\left(\frac{2}{3}(y-1)^3(y-1)^{-2}+2y(y-1)\right)\\ &= (18(y-1)^{-2}+6y)-\left(\frac{2}{3}(y-1)+2y^2-2y\right)\\ &= 18(y-1)^{-2}-2y^2+\frac{22}{3}y+\frac{2}{3},\end{aligned}$$

and then plug them in to the outer integrals:

$$\begin{aligned}\int_2^3\int_2^3 &(2x^2(y-1)^{-2}+2y)\mathrm{d}x\,\mathrm{d}y\\ &= \int_2^3\left(\frac{38}{3}(y-1)^{-2}+2y\right)\mathrm{d}y\\ &= \left(-\frac{38}{3}(y-1)^{-1}+y^2\right)\Big|_{y=2}^{y=3}\\ &= \left(-\frac{38}{3}((3)-1)^{-1}+(3)^2\right)-\left(-\frac{38}{3}((2)-1)^{-1}+(2)^2\right)\\ &= \left(-\frac{19}{3}+9\right)-\left(-\frac{38}{3}+4\right)=\frac{34}{3}.\end{aligned}$$

and

$$\begin{aligned}
&\int_3^4 \int_{y-1}^3 (2x^2(y-1)^{-2} + 2y)\mathrm{d}x\,\mathrm{d}y \\
&\qquad = \int_3^4 \Big(18(y-1)^{-2} - 2y^2 + \frac{22}{3}y + \frac{2}{3}\Big)\mathrm{d}y \\
&\qquad = \Big(-18(y-1)^{-1} - \frac{2}{3}y^3 + \frac{11}{3}y^2 + \frac{2}{3}y\Big)\Big|_{y=3}^{y=4} \\
&\qquad = \Big(-18((4)-1)^{-1} - \frac{2}{3}(4)^3 + \frac{11}{3}(4)^2 + \frac{2}{3}(4)\Big) \\
&\qquad\qquad - \Big(-18((3)-1)^{-1} - \frac{2}{3}(3)^3 + \frac{11}{3}(3)^2 + \frac{2}{3}(3)\Big) \\
&\qquad = \Big(-6 - \frac{128}{3} + \frac{176}{3} + \frac{8}{3}\Big) - (-9 - 18 + 33 + 2) \\
&\qquad = \frac{14}{3}.
\end{aligned}$$

The astonishing thing is that, though we did very different intermediate work along the way, we wind up with the same answer:

$$\int_2^3 \int_2^{x+1} (2x^2(y-1)^{-2} + 2y)\mathrm{d}y\,\mathrm{d}x = 16$$

and

$$\int_2^3 \int_2^3 (2x^2(y-1)^{-2} + 2y)\mathrm{d}x\,\mathrm{d}y + \int_3^4 \int_{y-1}^3 (2x^2(y-1)^{-2} + 2y)\mathrm{d}x\,\mathrm{d}y = \frac{34}{3} + \frac{14}{3} = 16.$$