In this handout, we will discuss the effect that derivatives and their sign diagrams have on the shape of a graph. There are a lot of sign diagrams to worry about, but it can be hard to keep track of which diagram means what. We'll use as our extended example the function from Exercise 12.2.5,

$$R(x) = -0.928x^3 + 31.492x^2 - 326.80x + 1143.88,$$

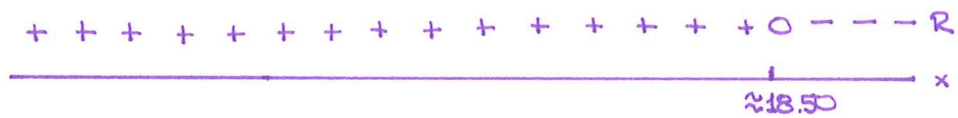
that measures the revenue (in billions of dollars) earned by an airline that flies $x$ hundred billion seat–miles *. We find that $R(x) = 0$ for exactly one value of $x$, satisfying $x \approx 18.50$ † Plugging in, say, $x = 18$ gives

$$R(18) = -0.928 \cdot 18^3 + 31.492 \cdot 18^2 - 326.80 \cdot 18 + 1143.88 = 52.792 > 0,$$

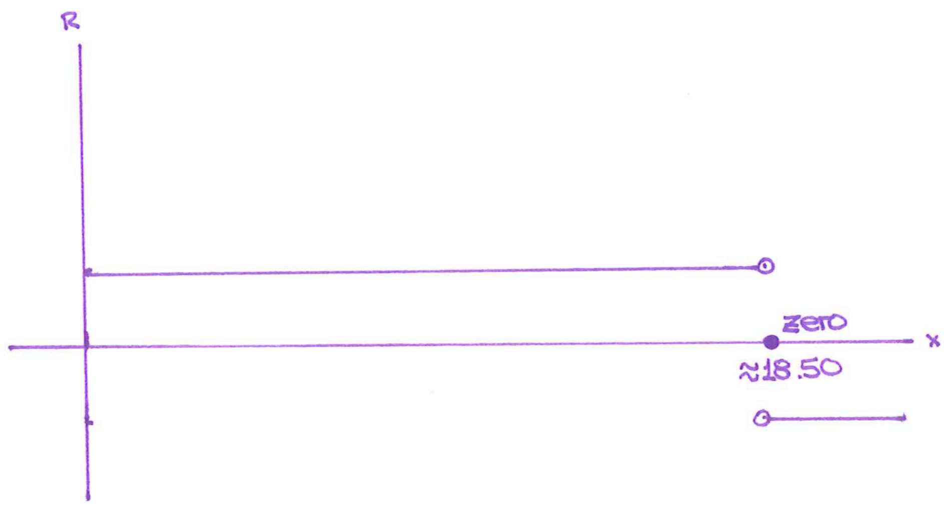
so $R > 0$ for $x < 18.50$. Plugging in, say, $x = 19$ gives

$$R(19) = -0.928 \cdot 19^3 + 31.492 \cdot 19^2 - 326.80 \cdot 19 + 1143.88 = -61.860 < 0$$

so $R < 0$ for $x > 18.50$. Thus, a sign diagram for the function $R$ looks like:




This tells us almost *nothing* about the graph of $R$, just when it's above and when it's below the $x$-axis:




* The term 'seat–mile' is unfamiliar, but it is a compound unit like, say, 'kilowatt–hour'. Just as a kilowatt–hour is a unit of energy corresponds to doing one kilowatt of work for one hour, a seat–mile corresponds to one seat on an airplane flying one mile. For example, a 500-mile airline flight on a 40-seat airplane would rack up 20,000 seat–miles.

The original problem only claims that this model is valid for $8 \leq x \leq 12$, but the *formula* makes sense for any $x$-value, even if it doesn't model the same thing; so we'll drop the restrictions on $x$.

† It's reasonable to ask where we got this value. One way would be just to graph the function and use the TRACE button on a TI calculator, or more sophisticated approaches such as the CALC > zero or MATH > Solver… tool; but we're trying to figure out how to sketch graphs ourselves, so that's not very satisfying. A brute-force way would be simply to enumerate many, many test values, until you got one that gave an output *close* to 0, and then plug in nearby values to try to find one that gave an output of *exactly* 0. There's no way to get around the requirement to make some initial guess, but there *is* a way to refine that guess intelligently; it's called Newton's method. We won't discuss it here, but it boils down to the idea of replacing solving a *cubic* equation by solving a *linear* equation—essentially, the tangent-line equation.

Surely this isn't the graph of $R$? No, it's absolutely not; but it's a reasonable first stab if we *only* know the sign diagram for $R$.

Part of the reason that the graph of $R$ above looks so crude is that it's not *changing*; it's just two horizontal lines, with a jump discontinuity in between. To take the change into account, we need the *derivative* of $R$:

$$R' = -2.784x^2 + 62.984x - 326.80.$$

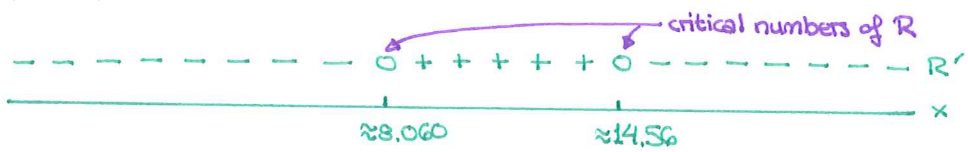
It's pretty much hopeless to try to factor this, so we use the quadratic formula to find its roots:

$$x = \frac{-62.94 \pm \sqrt{62.984^2 - 4(-2.784)(-326.80)}}{2(-2.784)} \approx 8.060, 14.56.$$

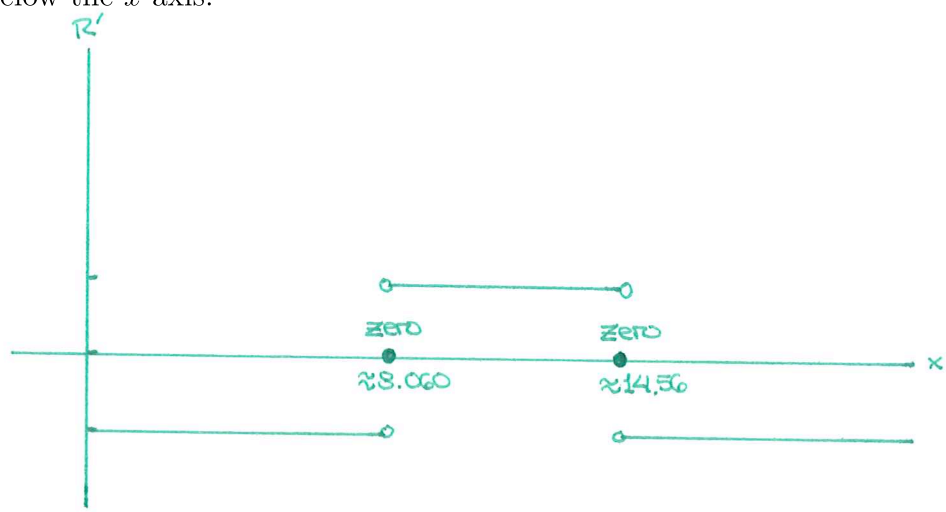
Plugging in, say, $x = 8$, $x = 9$, and $x = 15$ gives

$$R'(8) = -2.784 \cdot 8^2 + 62.984 \cdot 8 - 326.80 = -1.104 < 0,$$
$$R'(9) = -2.784 \cdot 9^2 + 62.984 \cdot 9 - 326.80 = 14.552 > 0,$$
$$R'(15) = -2.784 \cdot 15^2 + 62.984 \cdot 15 - 326.80 = -8.440 < 0,$$

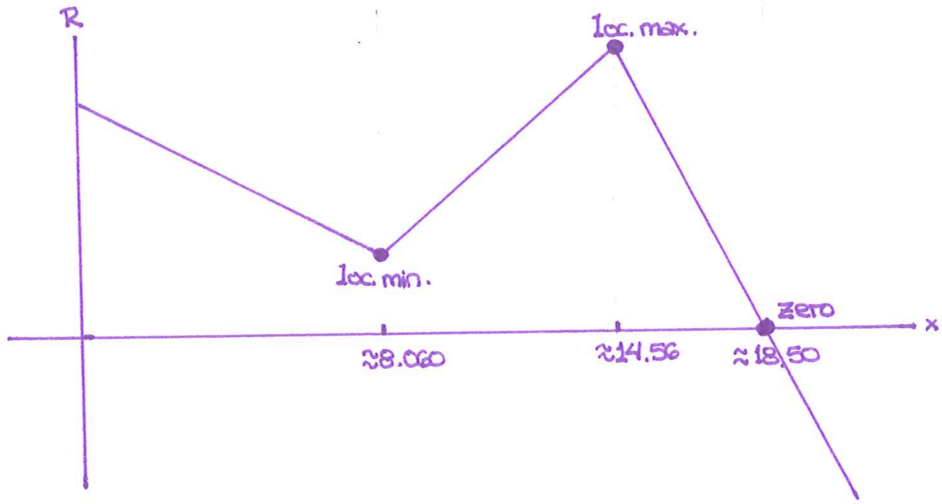
so that a sign diagram for the function $R'$ looks like:




Just like before, this tells us almost *nothing* about the graph of $R'$, just when it's above and when it's below the $x$-axis:




Again, this isn't actually the graph of $R'$, just a first approximation. Notice, however, that we're not *interested* in the graph of $R'$, but rather in the graph of $R$. We can change our original, very crude graph of $R$ so that it is increasing when it should be, and decreasing when it should be:




This still isn't the graph of $R$, but at least it's looking more like *some* kind of graph. In particular, we can now tell apart the different kinds of local extrema. Notice that the *zeroes* of $R'$ correspond to the *local extrema* of $R$. It's possible to have a zero of $R'$ which *isn't* a local extremum of $R$, if $R''$ doesn't change sign there; but this doesn't happen for us.

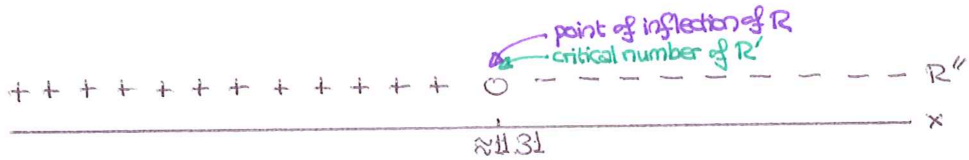
What's missing? Well, the bits of the graph of $R$ aren't really *lines*; their slopes are changing. To measure how the slopes are changing, we need the slope of the slope—the *second* derivative! We compute

$$R'' = -5.568x + 62.984,$$

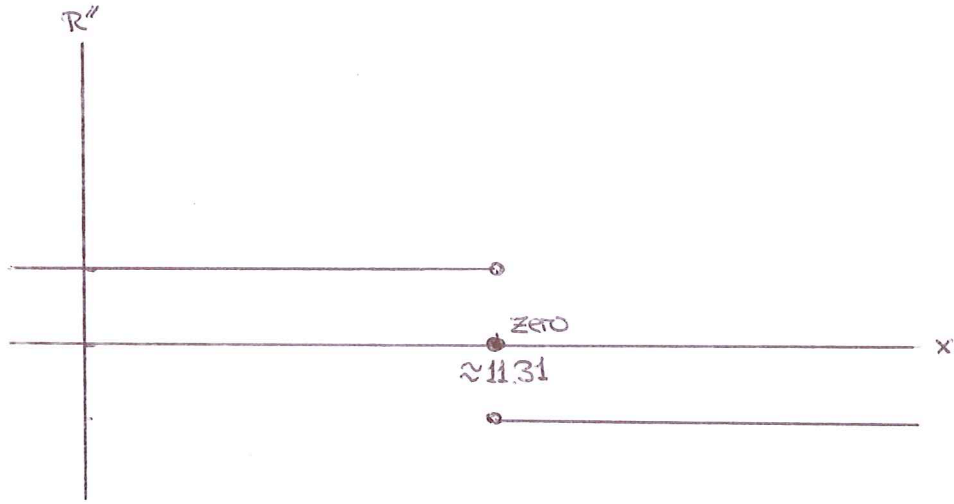
which is 0 when $x = \dfrac{62.984}{5.568} \approx 11.31$. Plugging in, say, $x = 11$ and $x = 12$ gives

$$R''(11) = -5.568 \cdot 11 + 62.984 = 1.736 > 0 \quad \text{and} \quad R''(12) = -3.832 < 0,$$

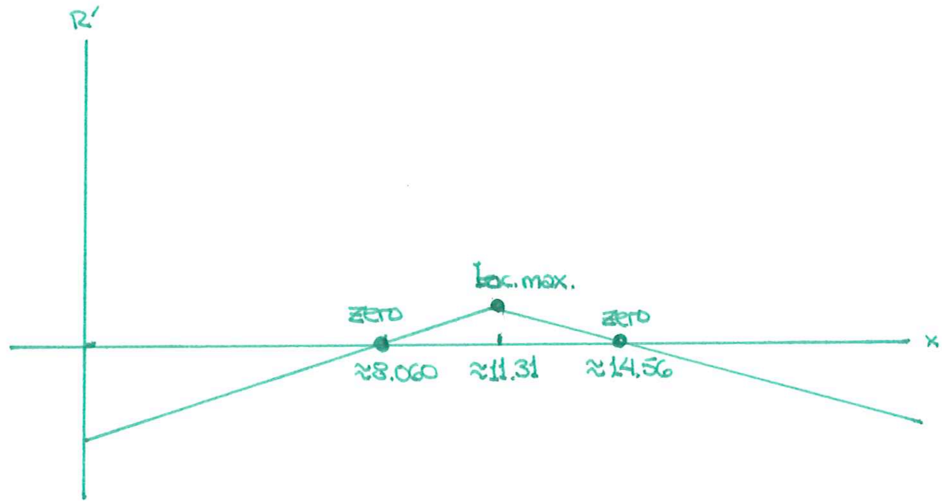
so that a sign diagram for the function $R''$ looks like:



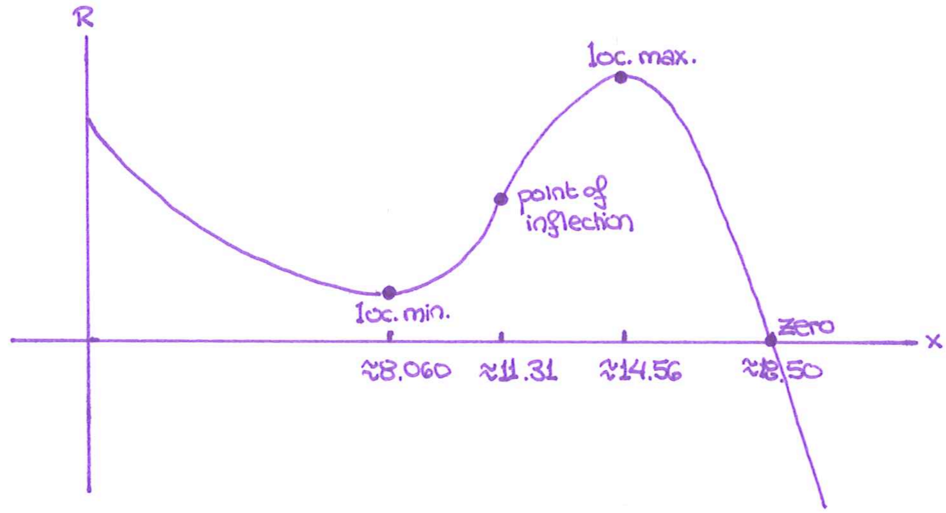

Just like before, this tells us almost *nothing* about the graph of $R''$, just when it's above and when it's below the $x$-axis:




Just as we used the graph of $R'$ to refine the graph of $R$, we can use the graph of $R''$ to refine the graph of $R'$ so that it is increasing when it should be, and decreasing when it should be:




Notice that the *zero* of $R''$ corresponds to the *local maximum* of the derivative $R'$. This *doesn't* mean that the original function $R$ reaches a local maximum there; instead, it means that that's where the original function $R$ will be (locally) steepest. In general, we should draw $R$ 'curling up' (in technical terms, *concave up*) when $R'$ is increasing, and 'curling down' (in technical terms, *concave down*) when $R'$ is decreasing:




The *zero* of $R''$ corresponds to the *local maximum* of $R'$, which corresponds to the *point of inflection* of $R$. It's possible to have a zero of $R''$ which *isn't* a point of inflection of $R$, if $R''$ doesn't change sign there; but this doesn't happen for us. Our picture of $R$ is now extremely accurate (except that we haven't paid any attention to hitting, say, the right $y$-intercept [‡]), but, if we wanted to be still *more* accurate, we could take still *more* derivatives. It gets harder and harder to see the effects of higher derivatives on the original graph, though.

By the way, in our work above, we called $x \approx 11.31$ a point of inflection for $R$, but the text calls it a *point of diminishing returns*. The terms mean the same thing; which one you use depends on what hat you're wearing. If you're wearing a mathematician's hat, then you notice that the graph has changed from 'curling up' to 'curling down', and you call it a point of inflection. If you're wearing an economist's or accountant's hat, then you notice that, although revenue is still increasing (since the first derivative $R'$ is still positive at $x \approx 11.31$; it's $R'(11.31) \approx 29.43$), it's doing so more slowly. Thus, the returns on an investment spent in increasing seat–miles aren't what they were in the early days, so perhaps that money might be better spent elsewhere.

[‡] Technically, we should call it an $R$-intercept, but that just sounds confusing.