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Classifying amenable operator algebras

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Abstract

These notes taken live in the lectures by Austin Shiner, with very light editing and verbal comments added by Stuart White. No responsibility is taken for any errors.

Day 1

The Classification Theorem

This all begins with the famous classification of injective von Neumann algebras.

Theorem 1.1.1 (Connes'76, Haagerup '85)**.** *Separably acting injective factors are classiified by their type and flow of weights*

Classification of C*-algebras starts with Glimm, Bratteli and Elliott's classification of AF algebras. But really starts in the very late 80's with Elliott's work on A$\mathbb{T}$ algebras. Today we have the following definitive theorem, which is a C*-algebra parallel of Theorem 1.1.1.

Theorem 1.1.2 (Many)**.** *Unital Simple Separable Nuclear (USSN) $\mathcal{Z}$-stable C*-algebras satisfying the UCT are classified by K-theory and traces.*[1] *Ie. for A, B USSN $\mathcal{Z}$-stable UCT, we have $A \cong B$ iff there exists isomorphisms $\alpha_i : K_i(A) \to K_i(B)$ and $\gamma : T(A) \to T(B)$ such that $\langle \gamma(T), x\rangle = \langle T, \alpha_0(x)\rangle$ for all $x \in K_0(A), \tau \in T(B)$, and $\alpha_0([1_A]) = [1_B] \in K_0(B)$.*

Here "Unital simple separable nuclear" corresponds to working with injective factors. The two other conditions: $\mathcal{Z}$-stability and satisfying the UCT will be explained later. Take them as black boxes for now. Algebras satisfying all these axioms are called *classifiable*.

There is a dichotomy between whether classifiable algebras have traces or not. When they do not, they are purely infinite, and the classification is the Kirchberg–Phillips theorem in the 90's. These lectures will focus on the case where there are traces. The methods are quite

[1] Here $T(A)$ is the collection of tracial states on A. It is a Choquet simplex.

different between the two sides of the dichotomy (though there is a recent new approach to the Kirchberg–Phillips, inspired by the stably finite methods presented here).

The main focus of these lectures is on unital C*-algebras. There are also definitive classification theorems for non-unital algebras, where traces are replaced by the cone of densely defined lower semicontinuous tracial weights, and instead of keeping track of the unit (which no longer exists), one keeps track of the scale; those elements in $K_0(A)$ realized by projections in the algebra.

Example 1.1.3. For (X, μ) a standard probability space (eg. $L^\infty[0,1]$). Suppose we have a countable discrete amenable group Γ and $\Gamma \curvearrowright (X, \mu)$ is measuring preserving, free and ergodic. Then $L^\infty(X, \mu) \rtimes \Gamma$ is an injective II_1 factor if it is infinite dimensional. Thus all these are isomorphic by Theorem 1.1.1.

Example 1.1.4. Suppose $\mathbb{Z} \curvearrowright \mathbb{T}$ by multiplication by $e^{2\pi i\theta}$ where θ is irrational. Then $L^\infty(\mathbb{T}) \rtimes \mathbb{Z}$ is independent of θ.

For C*-algebras, one also gets a huge class of examples from dynamics. The condition giving rise to simplicity is by now not so difficult, but the fact that all the examples below satisfy the UCT is a very deep theorem, proved by Tu, buidling on work of Higson and Kasparov on the Baum–Connes conjecture.

Example 1.1.5. For a compact metrizable space X and a countable discrete amenable group Γ and $\Gamma \curvearrowright X$ topologically free, minimal, then $C(X) \rtimes \Gamma$ is unital separable simple nuclear UCT.

There has been a huge body of work determining whether such algebras are $\mathcal{Z}$-stable with very rapid progress in recent years. Here is a major result (but note that this also holds for groups of subexponential growth which are not-elementary amenable). It could well hold for all amenable groups. The hypothesis of finite dimensionality of the space is necessary (by an example of Giol and Kerr).

Theorem 1.1.6 (Kerr-Naryshkin)**.** *If $\Gamma \curvearrowright X$ is free and $\dim(X) < \infty$ and Γ is elementary amenable, then $C(X) \rtimes \Gamma$ is $\mathcal{Z}$-stable.*

Definition 1.1.7 (KT_u)**.** The invariant $(K_0, K_1, T, [1], \langle , \rangle)$ is called KT_u.

Remark 1.1.8. The classical Elliot invariant $\mathrm{Ell}(A)$ includes K_0^+ but for USSA $\mathcal{Z}$-stable A we have $K_0^+(A) = \{x \in K_0(A) : \langle \tau, x\rangle > 0 \quad \text{for all } \tau \in T(A) \cup \{0\}\}$.

For classifiable C*-algebras we could equally use KT_u or Ell – they carry the same information – but we will use KT_u. This makes it more transparent that only the pairing is used, and the order structure on K_0 is not explicitly required. Also, the range of the invariant is much easier to describe for KT_u and fully understood.

Theorem 1.1.9 (Range of the Invariant)**.** *For all separable unital C*-algebra A, there exists USSN $\mathcal{Z}$-stable UCT C*-algebra B with $KT_u(A) \cong KT_u(B)$.*

One can also describe this abstractly. Any pair of countable abelian groups with a designated element as the class of the unit, together with any metrizable Choquet simplex and any possible pairing arises as the invariant of a classifiable C*-algebra (the only constraints are that in the finite case, the pairing sends the class of the unit to 1 under all traces, forcing this element to have no torsion).

This gives a strategy for obtaining results through classification. Build models with the required behavior and use classification. The following theorem is an excellent example, it is hard to imagine how this could be proved from the abstract hypotheses (just as it is hard to see how one would directly build a von Neumann algebraic Cartan subalgebra inside an injective factor, but this is straightforward using hyperfiniteness)

Theorem 1.1.10 (Li). *Every USSN $\mathcal{Z}$-stable UCT (classifiable) C*-algebra has a Cartan subalgebra and is isomorphic to a twisted groupoid C*-algebra.*

There are other examples of this technique, particularly for without traces where Spielberg showed certain Kirchberg algebras are semiprojective by building models where this can be seen directly. In a similar fashion he was able to lift certain group actions from the invariant to a Kirchberg algebra. This is much more developped for Kirchberg algebras where there are many more groupoid models currently known, but there is much potential to use stably finite classification to obtain further results of this nature.

Here is a concrete example – the C*-version of Example 1.1.4.

Example 1.1.11. Take the irrational rotation algebras $A_\theta = C(\mathbb{T}) \rtimes \mathbb{Z} = C^*(u,v)$. This is classifiable.

Given that these algebras are classifiable, what is the invariant? This is a place where the pairing between K-theory and traces plays an important role.

Remark 1.1.12. The K-theory of a crossed product $\alpha : \mathbb{Z} \curvearrowright A$ then Pimsner-Voiculescu gives a 6-term exact sequence

$$\begin{tikzcd}
K_0(A) \arrow[r, "1-\alpha_*"] & K_0(A) \arrow[r] & K_0(A \rtimes \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes \mathbb{Z}) \arrow[u] & K_1(A) \arrow[l] & K_1(A) \arrow[l, "1-\alpha_*"]
\end{tikzcd}$$

Using this, we have $K_0(A_\theta) \cong \mathbb{Z}^2 \cong K_1(A_\theta)$. In particular, $K_0(A)$ is generated by $1, p$ where $\tau(p) = \theta$. And A_θ has a unique trace τ coming from the Haar measure on $\mathbb{T}$. This is also easy to see. So neither the traces nor the K-theory distinguish these algebras. But the pairing does. The trace τ induces $\widehat{\tau} : K_0(A) \to \mathbb{R}$ with $\tau(1) = 1$ and $\tau(p) = \theta$ so that $\mathrm{Im}(\tau) = \mathbb{Z} + \theta\mathbb{Z}$. As a corollary

$$\mathcal{A}_{\theta_1} \cong A_{\theta_2} \Rightarrow \mathbb{Z} + \theta_1\mathbb{Z} = \mathbb{Z} + \theta_2\mathbb{Z} \Rightarrow \theta_1 = \pm\theta_2 \pmod{\mathbb{Z}}.$$

In fact the reverse implication also holds (without using classification). If $\theta_1 = \pm\theta_2 \pmod{\mathbb{Z}}$, then $A_{\theta_1} \cong A_{\theta_2}$ is easily checked (adjusting the angle by addition of an integer doesn't change the action, and changing the sign, is just rotation in the other direction, which is a conjugate action).

This calculation is now very old and allows one to see the angle of an irrational rotation C*-algebra upto integers and sign. What the classification theorem does is classify these algebras within a much larger class (this goes back to Elliott and Evans when they showed it was A$\mathbb{T}$, so within the class of A$\mathbb{T}$ algebras of real rank zero classified by Elliott. To see that A_θ is $\mathcal{Z}$-stable there are a range of techniques. Toms-Winter have a general result for $\mathbb{Z}$ actions on finite dimensional spaces, or one could use Elliott–Evans to get finite nuclear dimension. It was pointed out in the lecture that as these algebras have real rank zero and unique trace, one can use the order structure on K_0 to see that they have strict comparison, and hence are $\mathcal{Z}$-stable by a famous result of Matui and Sato. Also, Blackadar, Kumjian and Rørdam gave a relatively elementary argument coming from rational approximations to θ to prove almost divisibility, and hence obtain $\mathcal{Z}$-stability.

Let us now turn to the classification of maps. Going back to Murray and von Neumann's uniqueness of the hyperfinite II$_1$ factor, classification results for operator algebras have been obtained by first classifying maps.

The following result is a consequence of Connes' theorem (it would be straightforward for separable hyperfinite M).

Remark 1.1.13. If M is a separable injective Von Neumann algebra and N is a II_1 factor, then if $\varphi, \psi : M \to N$ are normal unital and $\tau_N \circ \varphi = \tau_N \circ \psi$ then there exists a sequence of unitaries (u_n) in $U(N)$ such that $u_n \varphi(a) u_n^* \to \psi_n(a)$ for any $a \in A$ in SOT.

Furthermore, for existence if $\tau_M \in T(M)$ is a normal trace then there exists a unital normal $\varphi : M \to N$ such that $\tau_N \circ \varphi = \tau_M$.

The corresponding C*-result is the following classification of embeddings. Outlining this will be the main objective of a number of the lectures.

Theorem 1.1.14 (Classification of embeddings, CGSTW). *For A unital separable nuclear UCT and B is unital simple $\mathcal{Z}$-stable and all quasitraces on B are traces (eg. if B is exact), then unital embeddings $A \hookrightarrow B$ are classified by the total invariant $\underline{K}T_u = (K_0, K_1, T, \overline{K_1}^{\mathrm{alg}}, K_0(-, \mathbb{Z}_n), [1], \textit{pairings})$.*

Here there are a number of natural pairing maps between all these objects (8, if one allows various infinite families to be packaged as one collection of pairing maps), and these form part of $\underline{K}T_u$.

The classification of embeddings theorem can be extended further by moving hypotheses from the algebras to morphisms. For example, one can take A to be exact and drop simplicity of B by requiring the map to by full and nuclear. i.e. the simplicity and amenability hypotheses can be placed on the maps. But, while we can classify maps in vast generality, when one symmeterises assumptions this will not lead to any larger classification of algebras.

Intertwining Arguments

How do we show $A \cong B$ for abstract C^*-algebras. Need *-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ with $\psi\varphi = \mathrm{id}_A$, $\varphi\psi = \mathrm{id}_B$. As analysts, we're trained not to try and prove things are equal on the nose, but rather that they are close. So can we relax equality here?

Definition 1.2.1 (Approximately Unitarily Equivalent)**.** Maps $\varphi, \psi : A \to B$ with B unital A separable are **approximately unitarily equivalent** if there exists a sequence of unitaries $(u_n)_{n=1}^\infty \subseteq B$ such that

$$\|u_n\varphi(a)u_n^* - \psi(a)\| \to 0 \quad \text{for all } a \in A$$

Written $\varphi \approx_u \psi$.

Theorem 1.2.2 (Elliott's two-sided approximate intertwining argument)**.** *If A, B are unital separable C^*-algebras, and we have $*$-homomorphisms $\varphi : A \to B, \psi : B \to A$ such that $\varphi \circ \psi \approx_u \mathrm{id}_B$ and $\psi \circ \varphi \approx_u \mathrm{id}_A$, then $A \cong B$. In fact there exists $\widehat{\varphi} : A \to B$ such that $\widehat{\varphi} \approx_u \varphi$ and $(\widehat{\varphi})^{-1} \approx_u \psi$.*

Proof. Set $\varphi_1 = \varphi$. Note that $\psi \circ \varphi_1 = \psi \circ \varphi \approx_u \mathrm{id}_A$, so for fixed $\mathcal{F}_1 \subseteq A$ finite and $\epsilon_1 > 0$ there exists $u_1 \in U(A)$ such that

$$\|u_1\psi(\varphi_1(a))u_1^* - a\| < \epsilon_1 \quad \text{for all } a \in \mathcal{F}_1$$

Then set $\psi_1 = \mathrm{Ad}(u_1)\psi$. Note that

$$\varphi \circ \psi_1 = \varphi \circ \mathrm{Ad}(u_1) \circ \psi = \mathrm{Ad}(\varphi(u_1)) \circ \varphi \circ \psi \approx_u \varphi \circ \psi \approx_u \mathrm{id}_B,$$

so for a fixed $\mathcal{G}_1 \subseteq B$ finite there exists $v_2 \in U(B)$ such that

$$\|v_2(\varphi(\psi_1(b))v_2^* - b\| < \epsilon_1 \quad \text{for all } b \in \mathcal{G}_1.$$

Then define $\varphi_2 = \mathrm{Ad}(v_2)\varphi : A \to B$. Then $\psi \circ \varphi_2 \approx_u \mathrm{id}_A$ so we get a $\psi_2 : B \to A$ as before and so on. We get a diagram which commutes approximately on larger and larger finite subsets and smaller and smaller ϵ

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_1"'] & & A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_2"] & & A \arrow[rr] \arrow[dr] & & \dots & \\
& B \arrow[ur, "\psi_1"'] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur, "\psi_2"] \arrow[rr, "\mathrm{id}_B"'] & & B \arrow[ur] \arrow[rr] & & \dots
\end{tikzcd}.$$

Note that

$$\varphi_{n+1} = \varphi_{n+1}\,\mathrm{id}_A \approx_u \varphi_{n+1}(\psi_n\varphi_n) = (\varphi_{n+1}\psi_n)\varphi_n \approx_u \mathrm{id}_B\,\varphi_n = \varphi_n$$

If we choose the $\mathcal{F}_n$ with $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $\overline{\bigcup_n \mathcal{F}_n} = A$ and same for the $\mathcal{G}_n$, and $\epsilon_1 > \epsilon_2 > \cdots > 0$ with $\sum \epsilon_n < \infty$, then what we get if we do this carefully is that $(\varphi_n(a))_n \subseteq B$ and $(\psi_n(b))_n \subseteq A$ are Cauchy for any $a \in A, b \in B$. Then define

$$\widehat{\varphi} : A \to B$$

$$\widehat{\varphi}(a) = \lim_n \varphi_n(a)$$

and similarly for $\widehat{\psi}$. These are $*$-homomorphisms and as one can check $\widehat{\psi}\widehat{\varphi} = \mathrm{id}_A$ and $\widehat{\varphi}\widehat{\psi} = \mathrm{id}_B$. $\square$

Corollary 1.2.3. *Suppose we have a functor $F : \mathsf{C^*Alg} \to \mathsf{C}$ into some category C such that F is $\approx_u$ invariant (ie. $\varphi \approx_u \psi$ implies $F\varphi = F\psi$). Suppose further that $\mathcal{S}$ is the class of unital separable C^*-algebras and assume*

1. *(Uniqueness) If $A, B \in \mathcal{S}$ and if $\varphi, \psi : A \to B$ with $F\varphi = F\psi$ then $\varphi \approx_u \psi$*
2. *(Existence) If $A, B \in \mathcal{S}$ and there is some morphism $\alpha : FA \to FB$, then there exists $\varphi : A \to B$ such that $F\varphi = \alpha$.*

Then, if $A, B \in \mathcal{S}$ and $FA \cong FB$, then $A \cong B$. In fact, if $FA \to FB$ is an isomorphism then it is induced by some isomorphism $A \to B$.

Proof. Suppose we have $A, B \in \mathcal{S}$ and an isomorphism $\alpha : FA \to FB$. Choose $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $F\varphi = \alpha, F\psi = \alpha^{-1}$. Now, $F(\psi\varphi) = F(\mathrm{id}_A)$ and $F(\varphi\psi) = F(\mathrm{id}_B)$ and then uniqueness implies $\psi\varphi \approx_u \mathrm{id}_A, \varphi\psi \approx_u \mathrm{id}_B$. Thus by the intertwining argument we get $\widehat{\varphi} : A \to B$ with $\widehat{\varphi} \approx_u \varphi$ and $(\widehat{\varphi})^{-1} \approx_u \psi$, so by $\approx_u$ invariance we get that $F\widehat{\varphi} = F\varphi = \alpha$. $\square$

Remark 1.2.4. Although it is not normally done in this way, one can set up the uniqueness of the hyperfinite II_1 factor with separable predual in a similar fashion. It is illustrative to note that these ideas do not work for type III factors, indeed there is not a unique hyperfinite type III factor.

Remark 1.2.5. Upto now one might have the impression the plan to prove the classification theorem is to apply Corollary 1.2.3 to KT_u. However if $\mathcal{S} = \{\text{classifiable C*-algebras}\}$ and $F = KT_u$ then existence holds, but uniqueness does not. More invariants will be needed to obtain uniqueness.

Example 1.2.6. Take $\mathcal{O}_3 \otimes \mathcal{O}_3$ and $\alpha \in \mathrm{Aut}(\mathcal{O}_3 \otimes \mathcal{O}_3)$ which flips: $\alpha(x \otimes y) = y \otimes x$. Then $K_0(\mathcal{O}_3 \otimes \mathcal{O}_3) = K_1(\mathcal{O}_3 \otimes \mathcal{O}_3) \cong \mathbb{Z}_2$ so $\alpha_* = \mathrm{id}_{K_*(\mathcal{O}_3)}$ but α is not approximately unitarily equivalent to the identity on $\mathcal{O}_3 \otimes \mathcal{O}_3$, for example because $\alpha \otimes \mathrm{id}_{\mathcal{O}_3}$ and $\mathrm{id}_{\mathcal{O}_3 \otimes \mathcal{O}_3 \otimes \mathcal{O}_3}$ differ on $K_0(\mathcal{O}_3 \otimes \mathcal{O}_3 \otimes \mathcal{O}_3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. However, this can be witnessed in $K_*(-, \mathbb{Z}_n) := K_*(- \otimes \mathcal{O}_{n+1}), n \geq 2$. It will be necesssary to add this 'K-theory with coefficients' to the invariant to obtain a uniqueness result.

Example 1.2.7. Take $A = \mathcal{Z} \wr \mathbb{Z} = \mathcal{Z}^{\otimes \mathbb{Z}} \rtimes \mathbb{Z}$ where the automorphism from $\mathbb{Z}$ shifts the indices. We know that $K_0(A) = K_1(A) = \mathbb{Z}$ and $K_0(A)$ is generated by $[1_A]$. We also know that A has unique trace and that $\mathrm{Aut}(KT_u(A)) \cong \mathbb{Z}_2$. We have a dual action $\gamma : \mathbb{T} \curvearrowright A$, and for $z_1, z_2 \in \mathbb{T}$ we have that $\gamma_{z_1}, \gamma_{z_2}$ are homotopic, hence $KT_u(\gamma_{z_1}) = KT_u(\gamma_{z_2})$, but a de la Harpe – Skandalis determinant argument shows that if $\gamma_{z_1} \approx_u \gamma_{z_2} \implies z_1 = z_2$. Indeed, if $z_1 \neq z_2$ then γ_{z_1} and γ_{z_2} do not agree on $\overline{K}_1^{\mathrm{alg}}(A) = \varinjlim U_n(A)/\overline{[U_n(A), U_n(A)]} \cong \mathbb{T} \oplus \mathbb{Z}_2$. In this case though, using that A has stable rank one, it turns out that $\overline{K}_1^{\mathrm{alg}}$ is just $U_1/\overline{[U_1(A), U_1(A)]}$.

This is why we extend the invariant for the classification of embeddings and work with an extended invariant $\underline{K}T_u$ rather than KT to obtain uniqueness. $\underline{K}T_u(A)$ is obtained by adding the total K-theory $\underline{K}(A)$ (consisting of all the groups $K_*(Z;\mathbb{Z}/n)$ together with the natural maps connecting them), and $\overline{K}_1^{\mathrm{alg}}(A)$ (together with further natural maps). However, adding additional data to prove uniqueness makes proving existence harder as one has to prove the existence of many more maps – all the allowed homomorphisms between $\underline{K}T_u(A) \to \underline{K}T_u(B)$.

Once one has the classification of embeddings, one gets a classification of the algebras in Theorem 1.1.2 from Elliott's two sided approximate intertwining argument. Notice that to apply this one would need to symmetrise assumptions, so that the collection of algebras classified is those A satisfying both the hypotheses on the domain, and on the codomain of the classification of embeddings, and also for which the identity map satisfies any required hypotheses on the map.[2] This will produce a classification as in Theorem 1.1.2 but with $\underline{K}T_u$ as the classifying invariant, rather than KT_u. A final detail is to show that any isomorphism $KT_u(A) \cong KT_u(B)$ gives rise to an isomorphism $\underline{K}T_u(A) \cong \underline{K}T_u(B)$.

Also, although we now have a strategy for proving Theorem 1.1.2 by proving existence and uniqueness theorems for morphisms $A \to B$, it is still very hard to produce a *-homomorphism between abstract C*-algebras A and B as required for the existence part of the classification of embeddings. Instead it is easier to produce approximate morphisms between C*-algebras. This should be compared with quasidiagonality: simple C*-algebras will never have non-zero maps into matrices, but they can have be quasidiagonal, i.e. have approximately multiplicative maps. Indeed the stably finite C*-algebras covered by the classification theorem are all quasidiagonal. This will be sketched during these lectures.

Definition 1.2.8 (Approximate Morphism)**.** An **approximate morphism** is a sequence $(\varphi_n)_n$ for ucp maps $\varphi_n : A \to B$ such that

$$\|\varphi_n(a_1a_2) - \varphi_n(a_1)\varphi_n(a_2)\| \to 0 \quad \text{for all } a_1, a_2 \in A.$$

When A is nuclear (so that we can use the Choi-Effros lifting theorem) this is also equivalent to a $*$-morphism $\varphi : A \to B_\infty = l^\infty(B)/c_0(B)$.

Remark 1.2.9. Being an approximate morphism without the ucp assumption is equivalent to a $*$-morphism $A \to B_\infty$.

Uniqueness of maps $A \to B_\infty$ immediately implies uniqueness of maps $A \to B$. There are simply less maps $A \to B$ than $A \to B_\infty$. However one needs both existence and uniqueness of maps $A \to B_\infty$ to run an Elliott intertwining argument to get existence of maps $A \to B$.

The main thrust of these lectures is devoted to the existence and uniqueness of maps $A \to B_\infty$ by $\underline{K}T_u$.

[2] This means that if one works with more general classfification of embedding theorems, with a separable exact domain, $\mathcal{Z}$-stable codomain and full nuclear maps, it is the requirement that the identity map be full and nuclear that forces the final classification to be for simple nuclear C*-algebras.

Day 2

KK-Theory

Remark 2.1.1. Throughout we'll assume A is separable.

Many very different looking pictures, and people don't really say how to move between them. We'll focus on the one most useful to us.

Remark 2.1.2. The goal of KK-theory is to force an abelian group structure on the set of $*$-morphisms mod some homotopy and other equivalence conditions we want.

Naively one approach that we might think would work is we could define

$$V(A,B) = \{A \to B \otimes K\}/\sim_h .$$

(Here K is the comparct operators on a separable infinite dimensional Hilbert space.) And we add $[\varphi], [\psi] : A \to B \otimes K$ by $[\varphi] + [\psi] = \left[\begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}\right]$ where we use a canonical isomorphism $K \cong M_2(K)$, which is unique up to homotopy.. Then $V(A,B)$ is an abelian semigroup (commutativity is through using rotation matrices to build a homotopy, and associativity in a similar fashion). It's not clear whether there is an additive identity. Then we can take the Grothendieck group of this semigroup: $V(A,B) - V(A,B)$.

If we then try with an example $A = \mathbb{C}$ we have $V(B) := V(\mathbb{C}, B) = P(B \otimes K)/\sim_h$ which is the Murray von Neumann semigroup of B. Then when B is unital we get back $V(B) - V(B) = K_0(B)$. But if B is non-unital, say $B = C(\mathbb{R}^2) \subseteq C(S^1)$, then $V(B) = 0$ while $K_0(B) = \mathbb{Z}$ generated by the difference of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\frac{1}{2}\begin{bmatrix} 1+x & y+iz \\ y-iz & 1-x \end{bmatrix} \in \mathcal{P}_\infty(C(S^2))$.

The basic problem is that there aren't enough projections to start with. In general there might not be enough *-homomorphisms $A \to B \otimes \mathcal{K}$. The solution is to allow for morphisms to take values in a much larger algebra.

Definition 2.1.3 (Cuntz Pair)**.** A **Cuntz pair** from A to B is a pair of $*$-morphisms

$$A \overset{\psi}{\underset{\varphi}{\rightrightarrows}} E \rhd B \otimes K$$

with $\operatorname{Im}(\varphi - \psi) \in B \otimes K$ where E is some C*-algebra containing $B \otimes K$ as an ideal. We think of such a Cuntz pair as a formal difference $\varphi - \psi$. $KK(A,B)$ will be defined as Cuntz pairs modulo a suitable notion of homotopy.

Notice that since we can take $E = M(B \otimes K)$ which contains a copy of $\mathcal{B}(H)$, as $B \otimes M(K) \subset M(B \otimes K)$, there are always maps $A \to M(B \otimes K)$.

Remark 2.1.4. Any $*$-morphism $\varphi : A \to B$ induces a Cuntz pair $A \overset{\varphi \otimes e_{11}}{\underset{0}{\rightrightarrows}} B \otimes K$.

Remark 2.1.5. Suppose
$$A \overset{\psi}{\underset{\varphi}{\rightrightarrows}} E \rhd B \otimes K$$
is a Cuntz pair. We always have a map $\lambda : E \to B \otimes K$ and this enables us to regard $(\lambda\varphi, \lambda\psi) \sim (\varphi, \psi)$. One can use this to only work with Cuntz pairs taking values in $M(B \otimes K)$ (allowing us to define homotopy between Cuntz pairs $A \rightrightarrows M(B \otimes K)$ as below), or alternatively view working with more general notions of homotopy coming from the Fredholm module picture of KK we have that $(\lambda\varphi, \lambda\psi)$ is homotopic to (φ, ψ)

Why allow for E, and not just insist Cuntz pairs take values in $M(B \otimes K)$ in the first place? The reason is that in natural examples (as happens when we construct a Cuntz pair in the classification of lifts) we obtain a Cuntz pair from some other ideal containing $B \otimes K$, rather than going directly into the multiplier algebra.

Definition 2.1.6 (Homotopy of Cuntz Pairs)**.** Suppose we have 2 Cuntz pairs $A \overset{\psi_0}{\underset{\varphi_0}{\rightrightarrows}} M(B \otimes K)$ and $A \overset{\psi_1}{\underset{\varphi_1}{\rightrightarrows}} M(B \otimes K)$. Then we write $(\varphi_0, \psi_0) \sim_h (\varphi_1, \psi_1)$ if there exists Cuntz pairs $A \overset{\psi_t}{\underset{\varphi_t}{\rightrightarrows}} M(B \otimes K)$, $0 \le t \le 1$ with $t \mapsto \varphi_t(a), t \mapsto \psi_t(a)$ strictly continuous,[3] and $t \mapsto \varphi_t(a) - \psi_t(a)$ is norm continuous for all $a \in A$.[4]

Definition 2.1.7. We define $KK(A, B)$ to be the set of all Cuntz pairs mod homotopy of Cuntz pairs

Remark 2.1.8. $KK(A, B)$ is an abelian group with $\oplus$, inverses are given by $-[\varphi, \psi] = [\psi, \varphi]$, and the zero element is $[\varphi, \varphi]$ for any $\varphi : A \to M(B \otimes K)$ (for instance we can take $\varphi = 0$; there is work to be done to show that all Cuntz pairs (φ, φ) are homotopic.).

One can define KK groups for non separable B, and $KK(A, B)$ is often defined for separable A and σ-unital B. One can extend B further, but separability of A is often crucial. One place is in defining the Kasparov product, through the usual use of separability in analysis to make countably many estimates and run $2^{-n}\epsilon$-arguments.

Remark 2.1.9. There are many $*$-homomorphisms $A \to M(B \otimes K)$ since $B(H) = M(K) \hookrightarrow M(B \otimes K)$. We can obtain a sufficiently generic such morphism using Voiculescu's theorem, and many computational aspects of KK go back to Voiculescu's theorem in some way.

Remark 2.1.10. Some properties of KK:

1. $KK(\mathbb{C}, B) \cong K_0(B)$ and $KK(C_0(\mathbb{R}), B) \cong K_1(B)$.
2. Also $KK(B, C_0(\mathbb{R}))$ gives BDF-theory, and $KK(B, \mathbb{C})$ gives BDF-theory of the suspension SB.

[3] The strict topology is given by $c_\lambda \to c$ strictly in $M(B \otimes \mathcal{K})$ iff $\|c_\lambda b - cb\|, \|c_\lambda^* b - c^* b\| \to 0$ for all $b \in B \otimes \mathcal{K}$.

[4] If one has Cuntz-pairs taking values in ideals $E_i \rhd M(B \otimes K)$,

3. $KK(A,-)$ is a covariant functor[5] and $KK(-,B)$ is a contravariant functor (this is easy: precompose a Cuntz pair by a map $C \to A$).

4. KK has various properties like homotopy invariance, stability, versions of the 6-term exact sequence etc. For instance, suppose we have a short exact sequence

$$0 \longrightarrow I \longrightarrow E \longrightarrow D \longrightarrow 0$$

where the quotient has a completely positive split. Then we get a 6-term exact sequence

$$\begin{tikzcd} KK(A,I) \arrow[r] & KK(A,E) \arrow[r] & KK(A,D) \arrow[d] \\ KK(A,SD) \arrow[u] & KK(A,SE) \arrow[l] & KK(A,SI) \arrow[l] \end{tikzcd}$$

Remark 2.1.11. Note that a map $C_0(\mathbb{R}) \to B$ induces a unital map $C(\mathbb{T}) \to \widetilde{B}$ and $C(\mathbb{T})$ is generated by a unitary so we get a unitary in $\widetilde{B}$. Taking differences of unitaries then gives $KK(C_0(\mathbb{R}), B) \cong K_1(B)$

Definition 2.1.12 (Kasparov Product)**.** Given separable A, B, C there exists an associative bilinear product (not to be defined) called the **Kasparov product**

$$\circ : KK(A,B) \times KK(B,C) \to KK(A,C)$$

where for $*$-homomorphisms $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ we have $[\psi] \circ [\varphi] = [\psi \circ \varphi]$.

Remark 2.1.13. Using the Kasparov product we get $KK(\mathbb{C}, A) \times KK(A,B) \to KK(\mathbb{C}, B)$ so any $\kappa \in KK(A,B)$ induces a group homomorphism $\kappa_0 : K_0(A) \to K_0(B)$ and similarly we get $\kappa_1 : K_1(A) \to K_1(B)$.

Remark 2.1.14. In this way we get an additive category KK whose objects consist of separable C*-algebras and whose morphisms are KK classes. Separable C*-algebras A and B are *KK-equivalent* if they are isomorphic in KK, i.e. if there exist elements $\kappa \in KK(A,B)$ and $\kappa' \in KK(B,A)$ with $\kappa' \circ \kappa = [\mathrm{id}_A]$ and $\kappa \circ \kappa' = [\mathrm{id}_B]$.

The UCT

Theorem 2.2.1 (The Universal Coefficient Theorem (Rosenberg, Schochett))**.** *Let A be a C*-algebra. Then TFAE:*

1. *For all C*-algebras B, there is a natural exact sequence*

$$0 \longrightarrow \operatorname{Ext}(K_*(A), K_{*+1}(B)) \longrightarrow KK(A,B) \xrightarrow{\kappa_0 \oplus \kappa_1} \operatorname{Hom}(K_*(A), K_*(B)) \longrightarrow 0\ ,$$

[5]This is harder to see in this picture, as not all maps $B \to C$ extend to maps $M(B \otimes \mathcal{K}) \to M(C \otimes \mathcal{K})$ but it can be done. This is a place where other pictures of KK are easier

2. *There exists LCH X such that $KK(A,-) \cong KK(C(X),-)$.*

Proof. (Sketch) For (2) implies (1), $KK(C_0(-), B)$ is a generalized homology theory of pointed compact spaces and then mimic the proof of Eilenberg and Maclane's UCT in algebraic topology. $\square$

Definition 2.2.2 (UCT)**.** We say A **satisfies the UCT** if for any B we have a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(K_*(A), K_{*+1}(B)) \longrightarrow KK(A,B) \longrightarrow \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0$$

Theorem 2.2.3 (Rosenberg, Schochett)**.** *If A, B satisfy the UCT then any isomorphism $\kappa_* : K_*(A) \to K_*(B)$ is induced by some invertible $\kappa \in KK(A,B)$.*

The map $KK(A,B) \to \mathrm{Hom}(K_*(A), K_*(B))$ above always exists: it is given by the Kasparov product as in Remark 2.1.13. The UCT asserts that this is surjective. The other map is defined in the other direction, from $\mathrm{Ker}(KK(A,B) \to \mathrm{Hom}(K_*(A), K_*(B)))$ back to $\mathrm{Ext}(K_*(A), K_{*+1}(B))$. It is a little hard to see in the Cuntz pair picture, but viewing an element in $KK(A,B)$ as given by an extension $0 \longrightarrow A \longrightarrow D \longrightarrow SB \longrightarrow 0$, the condition that such an extension lies in the kernel of $KK(A,B) \to \mathrm{Hom}(K_*(A), K_*(B))$ is precisely that the boundary maps of the extension vanish, and hence one gets an element of $\mathrm{Ext}(K_*(A), K_{*+1}(B))$. The UCT asserts that this map surjects onto $\mathrm{Ext}(K_*(A), K_{*+1}(B))$.

Example 2.2.4. The following have the UCT:

1. $C_0(X)$ for any LCH X,

2. If A_n have UCT then so does any inductive limit of the A_n with *injective connecting maps*.

3. If we have an extension $0 \longrightarrow I \longrightarrow E \longrightarrow D \longrightarrow 0$ and 2/3 of these have the UCT then so does the third. (Warning: it does not follow that quotients of C*-algebras with the UCT have the UCT. All cones $C_0(0,1] \otimes D$ have the UCT, but as in item 6 below, the quotient D might not).

4. If $A \sim B$ where $\sim$ is any of: homotopy equivalence, Morita equivalence, or KK-equivalence (there exists invertible $\kappa \in KK(A,B)$), then A has UCT iff B does.

5. If A satisfies UCT then so does $A \rtimes \mathbb{Z}$ and $A \rtimes \mathbb{R}$.

6. If A is UCT and Γ is torsion free amenable then $A \rtimes \Gamma$ is UCT. This is incredibly hard! It uses Higson and Kasparov's solution to the Baum-Connes conjecture, together with some hard homological algebra of Meyer and Nest.

7. If A, B are unital UCT then $A *_{\mathbb{C}} B$ is UCT. If ρ, γ are states on A, B respectively then $(A, \rho) * (B, \gamma)$ is UCT.

Slogan: if your algebra is built out of reasonable things then it will satisfy the UCT. For example, A$\mathbb{T}$-algebras, AH-algebras etc.

Remark 2.2.5 (Non example)**.** If Γ is an infinite biexact group with property (T). Then $C^*_\lambda(\Gamma)$ fails the UCT. This can be embedded in $\mathcal{O}_2$, so the UCT does not pass to subalgebras.

Remark 2.2.6. Major open problem: Do all separable nuclear C*-algebras satisfy the UCT? No conjectural counterexamples, and no approach for proving it either.

Theorem 2.2.7 (Tu/Barlak-Li, again building on Higson-Kasparov)**.** *If G is a locally compact hausdorff 2nd-countable étale amenable groupoid with twist Σ, then $C^*(G,\Sigma)$ satisfies the UCT.*

Remark 2.2.8. . For $\varphi, \psi : A \to B$ with $\varphi \approx_u \psi$ then it is possible for $[\varphi] \neq [\psi] \in KK(A,B)$. However, it is true that if there exists a continuous path $(u_t)_{t\geq 0} \subseteq \mathcal{U}(B)$ with $\lim_{t\to\infty} \|u_t\varphi(a)u_t^* - \psi(a)\| = 0$ for all $a \in A$ then $[\varphi] = [\psi]$.

Definition 2.2.9 (Topology on KK)**.** There is a natural topology on $KK(A,B)$ such that if $\varphi_n, \psi : A \to B$ are $*$-homomorphisms with $\|\varphi_n(a) - \psi(a)\| \to 0$ for all $a \in A$, then $[\varphi_n] \to [\psi]$ in $KK(A,B)$. Note that if we have $\varphi_n \to \varphi$ in point norm, then there exists a $*$-homomorphism $\Phi : A \to C(\mathbb{N}^+, B)$ (where $\mathbb{N}^+ = \mathbb{N} \cup \{\infty\}$ and $C(\mathbb{N}^+, B)$ are convergent sequences from $\mathbb{N}^+$ to B) with

$$\Phi(a)(n) = \begin{cases} \varphi_n(a) & \text{if } n < \infty \\ \varphi(a) & \text{if } n = \infty \end{cases}$$

For $\kappa_n, \kappa \in KK(A,B)$, we say $\kappa_n \to \kappa$ in $KK(A,B)$ iff there exists some $\widetilde{\kappa} \in KK(A, C(\mathbb{N}^+, B))$ such that

$$(\mathrm{ev}_n)_*(\widetilde{\kappa}) = \begin{cases} \kappa_n & \text{if } n < \infty \\ \kappa & \text{if } n = \infty \end{cases}$$

If we quotient $KK(A,B)$ by the closure of $\{0\}$, then we obtain $KL(A,B)$ which is invariant under approximate unitary equivalence, fixing the problem in the previous remark: if $\varphi, \psi : A \to B$, $\varphi \approx_u \psi$, then $[\varphi] = [\psi] \in KL(A,B)$.

And yes, KL is so named since L comes after K. Rørdam introduced KL assuming A satisfies the UCT, and this was extended by Dadarlat to a general definition.

Both philosophically and practically, one should think of KK as an invariant senstive (and potentially for classifying up to) assymptotic unitary equivalence (i.e. by continuous paths of unitaries), while KL works for approximate unitary equivalence (i.e. sequences of unitaries).

Theorem 2.2.10 (Dadarlat-Loring, Universal Multicoefficient theorem)**.** *If A satisfies the UCT, then for all B we have $KL(A,B) \cong \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ (just as with the UCT, the map $KL(A,B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ always exists, is natural, and defined using the Kasparov product) where $\underline{K}(A)$ is total K-theory (which consists of $K_*, K_*(-, \mathbb{Z}_n)$, and the change of coefficient maps)*

Non-Stable KK-Theory

Remark 2.3.1. There are 2 problems that concern us:

1. Given a Cuntz pair $A \overset{\psi}{\underset{\varphi}{\rightrightarrows}} M(B \otimes K)$ with $[\varphi, \psi] = 0$, then how are φ, ψ related?

2. Given $\kappa \in KK(A, B)$, when is κ induced by a $*$-morphism $A \to B$?

This should be compared with similar problems with K_0. Given projections $p, q \in A$ which agree in K_0, we know $p \oplus 1^{\oplus n} \sim q \oplus 1^{\oplus n}$, which we can view as a stable equivalence. However typically what we want is a cancellation result, implying that p and q are actually equivalent, without adding these copies of the unit.

In the case of Kirchberg algebras, these have very satisfactory answers.

Theorem 2.3.2 (Kirchberg-Phillips)**.** *If A is separable nuclear, and B is simple stable purely infinite, then*

$$KK(A, B) = \{\textit{embeddings } A \hookrightarrow B\}/ \sim_h .$$

(This homotopy can be implemented via strong asymptotic unitarily equvialence, i.e. a continuous path $(u_t)_{t=1}^{\infty}$ of unitaries with $u_1 = 1$ and $\|u_t\varphi(x)u_t^* - \psi(x)\| \to 0$ for all $x \in A$.) Also

$$KL(A, B) \cong \{\textit{embeddings } A \hookrightarrow B\}/ \approx_u .$$

In both cases, the unitaries are in the minimal unitisation of B

While the Kirchberg–Phillips theorem is often stated as a classification of UCT Kirchberg algebras by K-theory, the real statement is that Kirchberg algebras are classified by KL. The stable version of this statement is given below. There is a corresponding unital version, where one needs to work with KL-elements which preserve the unit.

Theorem 2.3.3. *If A, B are separable simple nuclear stable purely infinite (stable Kirchberg algebras), then $A \cong B$ iff $KL(A, -) \cong KL(B, -)$. If A, B satisfy the UCT, then $A \cong B$ iff $K_*(A) \cong K_*(B)$.*

Proof. Combine the previous Kirchberg-Phillips with an intertwining argument applied to the functor $\mathsf{C^*alg} \to \mathsf{KL}$. The role of the UCT in the second statement is to be able to lift any isomorphism $K_*(A) \cong K_*(B)$ to a KL-equivalence between A and B. $\square$

Returning to the general case, the Dadarlat–Eilers stable uniqueness theorem gives an answer to the first question in Remark 2.3.1. In the sprit of the stable equivalence of projections, one can go from $[\varphi, \psi] = 0$ to approximate unitary equivalence after adding on an additional morphism θ to both sides.

Theorem 2.3.4 (Dadarlat-Eilers, stable uniqueness Theorem)**.** *Suppose B is separable. Let $A \underset{\varphi}{\overset{\psi}{\rightrightarrows}} M(B \otimes K)$ be a Cuntz pair. Then $[\varphi, \psi] = 0$ iff there exists $\theta : A \to M(B \otimes K)$ and a continuous path $(u_t)_{t \geq 0} \subseteq \mathcal{U}(M_2\widetilde{(B \otimes K)})$ such that*

$$\left\| u_t \begin{pmatrix} \varphi(a) & 0 \\ 0 & \theta(a) \end{pmatrix} u_t^* - \begin{pmatrix} \psi(a) & 0 \\ 0 & \theta(a) \end{pmatrix} \right\| \to 0 \quad \textit{for all } a \in A$$

Moreover, θ can be taken to be any absorbing representation $\theta : A \to M(B \otimes K)$ (in particular for any $\nu : A \to M(B \otimes K)$, we have $\nu \oplus \theta \approx_u \theta$) (with unitaries in $M(B \otimes K)$).

It is tempting to ask why we can not just use absorption to get a result like this (and indeed this came up in the lectures). The point is that using absorption to get an approximate unitary equivalence would give unitaries coming from the muliplier algebra, and we really need that the unitaries in the Dadarlat-Eilers stable uniqueness theorem lie in the unitisation of the 2×2 matrices over $B \otimes K$, and not in the much larger algebra $M_2(M(B \otimes K)) \cong M(M_2(B \otimes K))$.[6] This will be crucial when we later use the stable uniqueness theorem. Indeed, we hinted earlier that we would obtain our Cuntz pairs $A \underset{\varphi}{\overset{\psi}{\rightrightarrows}} E \rhd B \otimes K$, and then obtain Cuntz pairs into the multiplier algebra as $A \underset{\lambda\varphi}{\overset{\lambda\psi}{\rightrightarrows}} M(B \otimes K) \rhd B \times K$. Suppose we could find unitaries in $(B \otimes K)^\sim$ approximately conjugating φ to $\lambda\psi$. Then since these can also be viewed as elements of E, they will also approximately conjugate φ to ψ in E. If our unitaries could only be found in $M(B \otimes K)$, then we would not learn anything about the original maps into the smaller algebra E.

[6]and note that M is being used for matrices and multipliers in this equation!)

Day 3

$\mathcal{Z}$-Stability and Regularity

Definition 3.1.1 ($\mathcal{Z}$-Stable)**.** A C*-algebra A is **$\mathcal{Z}$-stable** if $A \otimes \mathcal{Z} \cong A$ where the Jiang-Su algebra $\mathcal{Z}$ is the unique infinite-dimension classifiable C*-algebra with $KT_u(\mathcal{Z}) = KT_u(\mathbb{C})$.

The exact definition of $\mathcal{Z}$ is not particularly important (though we will sketch a definition below). What matters much more is the properties of $\mathcal{Z}$, and more importantly $\mathcal{Z}$-stable C*-algebras. There are also now a range of tools developed for testing for $\mathcal{Z}$-stability, without direct reference to $\mathcal{Z}$.

Remark 3.1.2. Note that if A is unital then $KT_u(A \otimes \mathcal{Z}) \cong KT_u(A)$.[7] However, one of the reasons we drop K_0^+ from the invariant is because K_0^+ does change under tensoring with $\mathcal{Z}$. For instance, one property of $\mathcal{Z}$-stable algebras is that if $nx > 0$ for some $n \geq 1$ then $x > 0$ and there are algebras that do not satisfy this.

Definition 3.1.3 (M_{2^∞}-Stable)**.** A is **M_{2^∞}-stable** if $A \otimes M_{2^\infty} \cong A$.

Proposition 3.1.4. *If A is separable unital, then A is M_{2^∞}-stable iff there exists an isomorphism $\theta : A \to M_2(A)$ and a sequence $(u_n)_{n=1}^\infty \subseteq M_2(A)$ of unitaries such that*

$$\left\| u_n \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} u_n^* - \theta(a) \right\| \to 0 \qquad \text{for all } a \in A$$

Proof. When $A = M_{2^\infty}$ we have $M_2(M_{2^\infty}) \cong M_{2^\infty}$, and any 2 unital $*$-homomorphisms of M_{2^∞} are approximately unitarily equivalent (due to Glimm). Next, in general if $\psi : A \xrightarrow{\cong} A \otimes M_{2^\infty}$ then choose an isomorphism $\theta_0 : M_2(M_{2^\infty}) \to M_{2^\infty}$. Then we have

$$A \xrightarrow{\psi} A \otimes M_{2^\infty} \xrightarrow{\mathrm{id}_A \otimes \theta} A \otimes M_2(M_{2^\infty}) = M_2(A \otimes M_{2^\infty}) \xrightarrow{M_2(\psi)^{-1}} M_2(A)$$

And $\mathrm{id}_A \otimes \theta$ is approximately unitarily equivalent to $\mathrm{id} \otimes d$ where $d(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, and composing everything we see that the above map $A \to M_2(A)$ is approximately unitarily equivalent to $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. The reverse direction is clear. $\square$

Remark 3.1.5. Note that, in the above proposition, the diagonal embedding $A \to M_2(A)$ is a point-norm limit of isomorphisms $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \lim_n \mathrm{Ad}(u_n^*)\theta(a)$.

This

[7] In general one can use the Kunneth formula to compute K-theory of tensor products (here as $\mathcal{Z}$ satisfies the UCT), but this is not necessary and one can show that $K_*(A) \cong K_*(A \otimes \mathcal{Z})$ directly from a definition of $\mathcal{Z}$. The identification of traces and pairing is also relatively straightforward.

Definition 3.1.6 (Strongly Self-Absorbing)**.** A C*-algebra A is **strongly self-absorbing** if A is unital, separable, infinite dimensional (this is here just to rule out $\mathbb{C}$), and there exists an isomorphism $\theta : A \to A \otimes A$ which is approximately unitarily equivalent to $a \mapsto a \otimes 1_A$.

Remark 3.1.7. At first glance it may seem like we need to specify what tensor product we are using. But if this isomorphism holds in any particular tensor norm, then the algebra must be nuclear, so the choice of tensor norm is irrelevant.

Theorem 3.1.8 (Effros and Rosenberg)**.** *Strongly self-absorbing algebras are simple, nuclear, and either unique trace or purely infinite.*

Example 3.1.9. The following are all strongly self-absorbing:

1. For a supernatural number n, the UHF-algebras of infinite type M_{n^∞}. This includes $\mathcal{Q}$ – the universal UHF algebra – corresponding to the supernatural number $\prod_{n\in\mathbb{N}} n$.

2. The Cuntz algebras $\mathcal{O}_n$ and $\mathcal{O}_\infty$,

3. The Jiang-Su algebra,

4. $M_{n^\infty} \otimes \mathcal{O}_\infty$, for a supernatural number n

Theorem 3.1.10. *The above examples are the only UCT strongly self-absorbing C*-algebras.*

Theorem 3.1.11. *If A is unital, separable, and D is ssa, then TFAE:*

1. $A \otimes D \cong A$,

2. *There exists* $\theta : A \to A \otimes D$ *with* $\theta \approx_u \mathrm{id}_A \otimes 1_D$,

3. *There is a unital embedding* $D \hookrightarrow A_\infty \cap A'$ *where* $A_\infty = l^\infty(A)/c_0(A)$.

Remark 3.1.12. The last condition above can be rewritten as a finitary statement using a presentation for D.

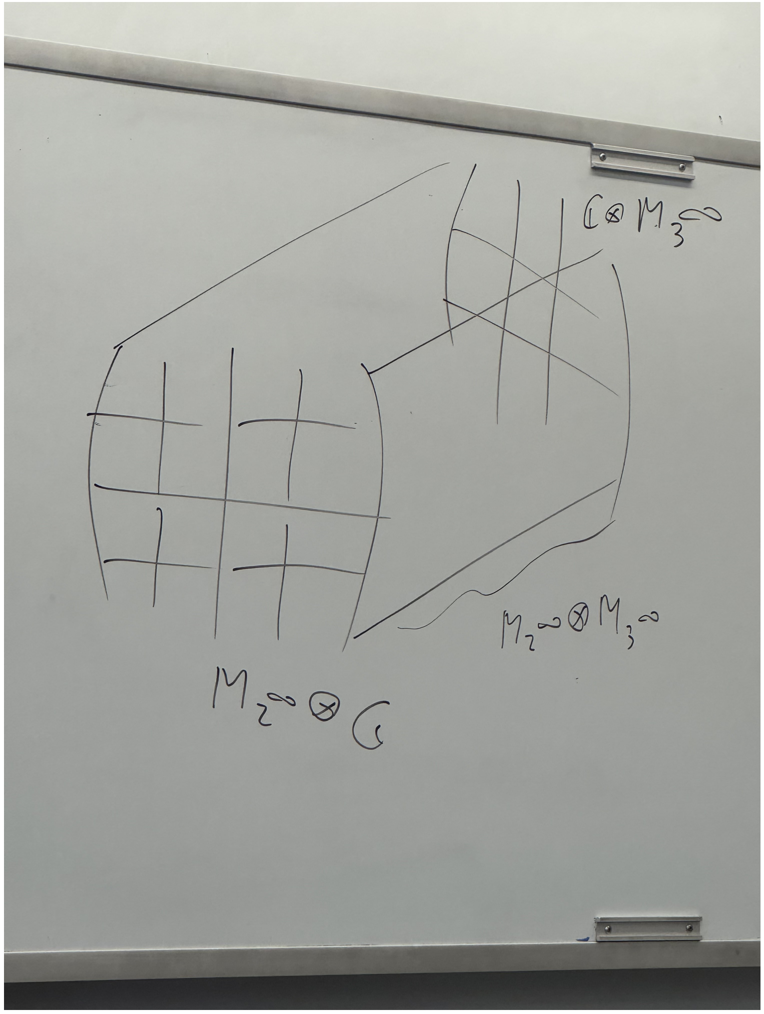
For example, $A \otimes M_{2^\infty} \cong A$ iff for any finite $F \subseteq A$ and $\epsilon > 0$ there exists unital $M_2 \hookrightarrow A$ such that $\|e_{ij}a - ae_{ij}\| < \epsilon$ for all $a \in A, i, j = 1, 2$.

Remark 3.1.13. The idea of defining $\mathcal{Z}$ is that we roughly want "$\mathcal{Z} = \bigcap_{n\geq 2} M_{n^\infty} = M_{2^\infty} \cap M_{3^\infty}$"

Commentary: At the level of K-theory, if we understand an abelian group after inverting 2 and after inverting 3, then we can recover the original group.

Definition 3.1.14. Define

$$\mathcal{Z}_{2^\infty,3^\infty} = \{f \in C([0,1], M_{2^\infty} \otimes M_{3^\infty}) : f(0) \in M_{2^\infty} \otimes \mathbb{C}, f(1) \in \mathbb{C} \otimes M_{3^\infty}\}.$$



Remark 3.1.15. Note that there are no non-trivial projections in $\mathcal{Z}_{2^\infty,3^\infty}$. Indeed if $p \in \mathcal{Z}_{2^\infty,3^\infty}$ then for $0 \leq t \leq 1$ we have $\mathrm{tr}(p(t)) \in \mathbb{Z}[1/6]$ (viewing $M_{2^\infty} \otimes M_{3^\infty} \subseteq M_{6^\infty}$), so $\mathrm{tr}(p(t))$ is constant as $\mathbb{Z}[1/6]$ is totally disconnected. However, $\mathrm{tr}(p(0)) \in \mathbb{Z}[1/2]$ and $\mathrm{tr}(p(1)) \in \mathbb{Z}[1/3]$ so $\mathrm{tr}(p(t)) \in \mathbb{Z}$ and hence $p(t) = 0$ or $p(t) = 1$ identically since it is a projection.

The following can be obtained from a short exact sequence argument using

$$0 \longrightarrow C_0((0,1), M_{2^\infty} \otimes M_{3^\infty} \longrightarrow \mathcal{Z}_{2^\infty,3^\infty} \longrightarrow M_{2^\infty} \oplus M_{3^\infty} \longrightarrow 0\ . \tag{1}$$

Theorem 3.1.16. $K_0(\mathcal{Z}_{2^\infty,3^\infty}) \cong \mathbb{Z}, K_1(\mathcal{Z}_{2^\infty,3^\infty}) = 0$.

Theorem 3.1.17 (Rørdam-Winter/Schemaitat)**.** *There exists (unique up to approximate unitary equivalence) $\alpha \in \mathrm{End}(\mathcal{Z}_{2^\infty,3^\infty})$ such that for any $\tau \in T(\mathcal{Z}_{2^\infty,3^\infty})$, we have $\tau\alpha = \tau_{\mathrm{leb}}$, and $\mathcal{Z} = \varinjlim(\mathcal{Z}_{2^\infty,3^\infty} \xrightarrow{\alpha} \mathcal{Z}_{2^\infty,3^\infty} \xrightarrow{\alpha} \cdots)$.*

The idea is that one views α as collapsing the interval as shown in the
SECOND DIAGRAM

Then one uses the standard way to turn an endormorphism on a C*-algebra into an automorphism on a containing algebra via a stationary inductive limit. The description of $\mathcal{Z}$ in Theorem 3.1.17 is not the original (which was in terms of a carefully defined inductive limit of similar dimension drop algebras $\mathcal{Z}_{p_n,q_n}$ for coprime integers (rather than supernatural numbers) p_n and q_n). Today for many purposes we can the conclusion of Theorem 3.1.17 as the definition of $\mathcal{Z}$.

It's worth noting that Rørdam-Winter's proof that an α as in Theorem 3.1.17 exists goes through the earlier construction of $\mathcal{Z}$; they build maps $\mathcal{Z}_{2^\infty,3^\infty} \to \mathcal{Z} \to \mathcal{Z}_{2^\infty,3^\infty}$ (which will certainly collapse trace).[8] Schemaitat uses classification results for morphisms between UHF-algebras (upto asymptotic unitary equivalence) to obtain an α through an intertwining argument. He gives a direct proof that $\mathcal{Z}$ is strongly self-absorbing from the inductive limit in Theorem 3.1.17 using asymptotic classification results for UHF-algebras.

Often the way to prove results for $\mathcal{Z}$-stable algebras, is to first understand how to prove the result in UHF-stable algebras, and then try and use that $\mathcal{Z}_{2^\infty,3^\infty}$ is locally UHF in a suitable sense.

Theorem 3.1.18 (Jiang)**.** *If A is unital $\mathcal{Z}$-stable, then $K_1(A) = \mathcal{U}(A)/\mathcal{U}_0(A)$*

This was first proved by Jiang from the original inductive limit picture. As part of the classification paper, part of this was reproved using the corresponding results for UHF-stable C*-algebras (and tensoring the short exact sequence (1) by A) in the classification paper, and the other part by Hua. These give very good examples of how to go from UHF-stable results to $\mathcal{Z}$-stable results.

Trace–kernel extension

Remark 3.2.1. Goal: A, B unital separable simple nuclear (ussn), $\mathcal{Z}$-stable UCT [not all these hypotheses are needed on both A and B]. Then unital $A \to B$ are classified up to $\approx_u$ by

$$\underline{K}T_u = (\underline{K}, \overline{K}^{\mathrm{alg}}, \mathrm{Aff}\, T, \text{compatibility})$$

Via intertwining arguments it's enough to classify embeddings $A \to B_\infty$ by $\underline{K}T_u$. (With some care one can replace B_∞ by B_ω; in the end it makes no real difference, intertwining is a bit easier with sequence algebras, traces are slightly nicer with ultrapowers. In the lectures we'll use what ever is most convenient at the time).

What can be said about B_∞ (or B_ω)? For example, B_∞ is not nuclear and is not separable. Also even if B is simple, B_ω is typically no longer simple (it will be when B is Kirchberg).

Example 3.2.2. $\prod_\omega M_n$, is not a simple C*-algebra:

$$\{(x_n)_{n=1}^\infty \in \prod M_n : \frac{\mathrm{rank}(x_n)}{n} = 0\} \lhd \prod_\omega M_n$$

[8] They don't show that $\tau\alpha = \tau_{\mathrm{leb}}$ but instead realise $\mathcal{Z}$ as the inductive limit for any trace collapsing α.

In fact if B_ω is simple, then either B is a matrix algebra, or B is simple and purely infinite, when B_ω is also simple and purely infinite.

Remark 3.2.3. For the rest of this section we'll assume $T(B) \neq 0$. Recall that classification is split into the traceless (purely infinite) case, and the stably finite case (where B has at least one trace).

Definition 3.2.4. If τ is a trace on B then we define

$$J_\tau = \{(b_n)_{n=1}^\infty \subseteq B : \lim_n \tau(b_n^* b_n)^{1/2} = 0\} \trianglelefteq B_\omega$$

We then define

$$J_B = \{(b_n)_{n=1}^\infty \in B_\omega : \lim_{n\to\omega} \sup_{\tau \in T(B)} \tau(b_n^* b_n)^{1/2} = 0\} \trianglelefteq B_\omega$$

We call J_B the trace-kernel ideal.

When computing K-theory one of the major tools is 6-term exact sequences, so it can be difficult to calculate K-theory for a simple C*-algebra, where have no ideals to cut it into smaller pieces. For B_ω the ideal J_B provides a tool for 6-term exact sequences, and importantly the quotient is well behaved.

Theorem 3.2.5. *Assume B is unital unique trace. Then $B^\omega := B_\omega/J_B$ (same as the 2-norm tracial ultrapower of B) is isomorphic to $(\pi_\tau(B)'')^\omega$. In particular, B^ω is a von Neumann algebra (in fact a finite factor).*

Proof. $\pi_\tau(B)''$ is a factor since τ is the unique normal trace. The map $\pi_\tau : B \to \pi_\tau(B)''$ induces a map $\pi_\tau^\omega : B^\omega \to (\pi_\tau(B)'')^\omega$. Injectivity is easy and surjectivity comes from Kaplansky density. □

Definition 3.2.6 (Trace-Kernel Extension)**.** We call the extension

$$e_B :\ 0 \longrightarrow J_B \xrightarrow{\iota} B_\omega \xrightarrow{q_B} B^\omega \longrightarrow 0$$

the **trace-kernel extension** of B.

Our strategy will be to split the classification into two parts. In the unique trace case, we understand the von Neumann algebra B^ω very well, and so can classify maps into this, up to an error lying in J_B.

Remark 3.2.7. Surprisingly, the trace-kernel extension was first used by Matui–Sato in the more complicated central sequence form

$$0 \longrightarrow J_B \cap B' \longrightarrow B_\omega \cap B' \longrightarrow B^\omega \cap B' \longrightarrow 0$$

(which is harder to show is short exact!) and they obtained a lot of information about $J_B \cap B'$, enabling them to get $\mathcal{Z}$-stability from strict comparison (for simple nuclear B with unique trace). One can view their strategy as a lifting problem: when B is nuclear with unique trace, Connes' theorem shows that $\pi_\tau(B)''$ is McDuff, and so $\mathcal{R}$ embeds into $B^\omega \cap B'$. $\mathcal{Z}$ is dense in $\mathcal{R}$, and the challenge is to lift the copy of $\mathcal{Z}$ back to $B_\omega \cap B'$:

$$\begin{tikzcd}
0 \arrow[r] & J_B \cap B' \arrow[r] & B_\omega \cap B' \arrow[r, "q_B"'] & B^\omega \cap B' \arrow[r] & 0 \\
 & & \mathcal{Z} \arrow[u, dashed, "?"] \arrow[r, hook] & R \arrow[u, hook] &
\end{tikzcd}$$

The use of the trace-kernel extension without central sequences came later.

Remark 3.2.8. The strategy for classifying maps $A \to B_\omega$:

1. Classify embeddings $A \to B^\omega$ (by traces)

2. Classify lifts along q_B:

$$\begin{tikzcd}
 & & & A \arrow[dl, dashed, "?"'] \arrow[d, "\theta"] & \\
0 \arrow[r] & J_B \arrow[r] & B_\omega \arrow[r, "q_B"'] & B^\omega \arrow[r] & 0
\end{tikzcd}$$

   The obstruction to existence of lifts is found in $KK^1(A, B_\omega)$. For uniqueness, given two lifts φ, ψ of θ, then (φ, ψ) forms a (A, J_B)-cuntz pair and so a class in $KK(A, J_B)$; this class will determine whether φ and ψ are approximately unitarily conjugate.

3. Compute $KK^*(A, J_B)$ in terms of things related to A, B (this is where UCT enters).

Remark 3.2.9. Here are more details for step 1 when B has a unique trace: $T(B) = \{\tau_B\}$. Since we assume B, and hence $\pi_{\tau_B}(B)''$ is infinite dimensional, so B^ω is a II_1 factor. Next assume A is separable nuclear. Due to Connes we have existence and uniqueness:

For uniqueness: if $\varphi, \psi : A \to B^\omega$ are $*$-homomorphisms with $\tau_{B^\omega}\varphi = \tau_{B^\omega}\psi$, then $\varphi \sim_u \psi$ (normally we'd have $\varphi \approx_u \psi$, but since we are in an ultrapower we can reindex and get unitary conjugation on the nose). Indeed define $\tau := \tau_{B^\omega}\varphi = \tau_{B^\omega}\psi$, so φ, ψ extend to maps $\overline{\varphi}, \overline{\psi} : \pi_\tau(A)'' \to B^\omega$, and $\pi_\tau(A)''$ is hyperfinite by Connes' theorem. This is a point in which it matters we can get internal structure on A. Then the result follows from Murray and von Neumann's classification of projections in a II_1 factor by their trace, which gives a classification of maps out of finite dimensional algebras into II_1 factors by trace. In this way we have $\overline{\varphi} \approx_u \overline{\psi}$. The approximate unitary equivalence here is in strong*-topology, but because we work in a ultrapower, one can reindex and get unitary equivalence of $\overline{\varphi}$ and $\overline{\psi}$, and hence unitary equivalence of φ and ψ.

For existence, if $\tau \in T(A)$, then there exists $\varphi : A \to B^\omega$ with $\tau_{B^\omega}\varphi = \tau_A$ by using amenability of the trace τ_A to get approximately multiplicative maps from A into matrix algebras which induce this trace, and embedding these matrix algebras into B^ω.

Day 4

Classifying Lifts

Remark 4.1.1. Let B be unital with $T(B) \neq 0$. Recall

$$B_\omega = l^\infty(B)/\{(b_n) : \lim_{n\to\omega} \|b_n\| = 0\}$$

$$B^\omega = l^\infty(B)/\{(b_n) : \lim_{n\to\omega} \sup_{\tau\in T(B)} \tau(b_n^* b_n)^{1/2} = 0\}$$

This gives the trace-kernel extension

$$0 \longrightarrow J_B \xrightarrow{\iota_B} B_\omega \xrightarrow{q_B} B^\omega \longrightarrow 0$$

Theorem 4.1.2 (Ozawa)**.** *Let B be simple exact and $\mathcal{Z}$-stable. Then $T(B^\omega) = T(B_\omega)$. Moreover the traces on B_ω are generated by the limit traces[9] coming from B.*

As described in the last lecture, when $T(B) = \{\tau\}$ one has classification into B^ω by traces as a consequence of Connes' theorem. When there are more traces, $\mathcal{Z}$-stability (in fact the weaker condition of uniform property Γ suffices), can be used to obtain von Neumann like behavior on B^ω (which is no longer a von Neumann algebra), and glue together the classification along the traces one gets applying from Connes' theorem into the ultrapower for each individual trace to a global result. This gives:

Theorem 4.1.3 (Castillejos-Evington-Tikuisis-White-Winter[10])**.** *Let A be separable unital nuclear, B unital $\mathcal{Z}$-stable, $T(B) \neq 0$. Then*

$$\{\textit{Unital } *\textit{-homs } A \to B^\omega\}/\sim_u \; = \{\textit{Unital positive } \operatorname{Aff} T(A) \to \operatorname{Aff} T(B_\omega)\}$$

Definition 4.1.4 (Full)**.** A map $\theta : A \to D$ is **full** if for any $a \in A\backslash\{0\}$, we have $\overline{D\theta(a)D} = D$. Equivalently, $\ker\theta = 0$ and $\theta(A) \cap D_0 = 0$ for any proper ideal $D_0 \supsetneq D$.

This definition should be compared with the condition in Voiculescu's theorem that we work with injective essential representations $A \to B(H)$, i.e. injective maps range does not intersect the compacts non-trivially. Fullness is the analogue for a map into an arbitrary C*-algebra.

Theorem 4.1.5 (Classification of Lifts, CGSTW)**.** *Let E be unital, separable, $\mathcal{Z}$-stable, let I be stable, and let A be separable, unital, nuclear, and let $\theta : A \to D$ be unital and full.*

[9] A limit trace on B_ω is a trace τ given by a sequence (τ_n) from $T(B)$, and $\tau((b_n)) = \lim_{n\to\omega} \tau_n(b_n)$. Then Ozawa's theorem says that $T(B_\omega)$ is the closure of the limit traces (noting that the collection of limit traces is already convex).

[10] CETWW proved this when B is also nuclear; the version of the result given here uses results from Carrión, Castillejos, Evington, Gabe, Schafhauser, White, Winter.

Consider an extension[11]

$$\begin{tikzcd} & & & A \arrow[d, "\theta"] & \\ 0 \arrow[r] & I \arrow[r, "\iota"] & E \arrow[r, "q"] & D \arrow[r] & 0 \end{tikzcd}$$

Then

1. *There exists a unital lift $\psi : A \to E$, ie. $q\psi = \theta$, iff there exists $\kappa \in KK(A, E)$ such that $[q]\kappa = [\theta] \in KK(A, D)$ and $\kappa_0[1_A] = [1_E] \in K_0(E)$.*
2. *For a fixed ψ as in (1),*

$$\{\textit{Unital } \varphi : A \to E : q\varphi = \theta\}/ \approx_u \textit{ with unitaries from } \widetilde{I}$$

is in bijective correspondence with

$$\{\kappa \in KL(A, I) : \kappa_0[1_A] = 0 \in K_0(I)\},$$

by the map

$$\varphi \mapsto [\varphi, \psi].$$

The point here is that given another lift φ of θ, then

$$A \underset{\varphi}{\overset{\psi}{\rightrightarrows}} E \rhd I$$

is a Cuntz pair so defines a class $[\varphi, \psi] \in KL(A, I)$.

Proof. (Sketch of injectivity for (2)): Assume we have a pair of maps $A \underset{\varphi}{\overset{\psi}{\rightrightarrows}} E$ with $q\varphi = q\psi = \theta$ (which is full) and $[\varphi, \psi] = 0 \in KL(A, I)$. Then we get the following diagram with exact rows

$$\begin{tikzcd} & & A \arrow[d, bend right, "\varphi"'] \arrow[d, bend left, "\psi"] & & \\ 0 \arrow[r] & I \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "\lambda"] & D \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I \arrow[r] & M(I) \arrow[r] & Q(I) \arrow[r] & 0. \end{tikzcd}$$

Then we have a Cuntz pair $A \underset{\lambda\psi}{\overset{\lambda\varphi}{\rightrightarrows}} M(I) \trianglerighteq I$ vanishing in $KL(A, I)$. Then by (a KL-version of the) Dadarlat-Eilers stable uniqueness theorem for any unitally absorbing $\mu : A \to M(I)$, there is a sequence $(u_n)_{n=1}^{\infty} \subseteq \mathcal{U}_2(\widetilde{I})$ of unitaries such that

$$u_n \begin{pmatrix} \lambda\varphi(a) & 0 \\ 0 & \mu(a) \end{pmatrix} u_n^* \to \begin{pmatrix} \lambda\psi(a) & 0 \\ 0 & \mu(a) \end{pmatrix} \quad \text{for all } a \in A.$$

[11]Correction: one shoud also assume ind : $K_1(D) \to K_0(I)$ vanishes. This holds when in the trace-kernel extension setting since $K_1(B^\omega) = 0$.

By the theorem below we know that $\lambda\varphi, \lambda\psi : A \to M(I)$ are unitally absorbing. Thus we can apply this twice to get:

$$\begin{pmatrix} \lambda\varphi & 0 \\ 0 & \lambda\varphi \end{pmatrix} \underbrace{\approx_u}_{\text{in } \mathcal{U}_2(\widetilde{I})} \begin{pmatrix} \lambda\psi & 0 \\ 0 & \lambda\varphi \end{pmatrix} \underbrace{\approx_u}_{\text{in } \mathcal{U}_2(\widetilde{I})} \begin{pmatrix} \lambda\psi & 0 \\ 0 & \lambda\psi \end{pmatrix}.$$

It is vital that these approximate unitary equivalences happen with unitaries in $U_2(\widetilde{I})$ so that they can be found in E. If we just used absorption, we would get $\lambda\varphi$ is approximately unitarily equivalent to $\lambda\psi$ with unitaries coming from $M(I)$, but there is absolutely no reason why this should give information about approximate unitary equivalence in E.

Thus we have that if $\varphi, \psi : A \to E$ are such that $q\varphi = q\psi = \theta$ and $[\varphi, \psi] = 0$, then $\varphi \oplus \varphi \approx_u \psi \oplus \psi$. Note that if $E \otimes M_{2^\infty} \approx E$ then it would follows that $\varphi \approx_u \psi$ (in the spirit of the last lecture. The approach to use $\mathcal{Z}$-stability for something is to prove it first for M_{2^∞} and M_{3^∞} and try to "glue it together" to get $\mathcal{Z}$). Similarly if E is M_{3^∞}-stable then $\varphi \oplus \varphi \oplus \varphi \approx_u \psi \oplus \psi \oplus \psi$ would imply $\varphi \approx_u \psi$. The rough idea of the rest of the proof (for which the details are omitted) is to patch these 2 together in $\mathcal{Z}_{2^\infty, 3^\infty}$ to get $\varphi \approx_u \psi$. □

Theorem 4.1.6 (Elliott-Kucerovsky, Kuc-Ng, Ortega-Perera-Rørdam)**.** *If A is separable unital nuclear, I-stable, $\mathcal{Z}$-stable, and $\mu : A \to M(I)$ is unital and full, then μ is unitally absorbing.*

Remark 4.1.7. Recall a unital $\mu : A \to M(I)$ is unitally absorbing if for all unital $\eta : A \to M(I)$ we have $\mu \oplus \eta \approx_u \mu$. Here the unitaries witnessing this approximate unitary equivalence are in $M(I)$.

Definition 4.1.8 (Separably $\mathcal{Z}$-Stable)**.** A C*-algebra A is **separably $\mathcal{Z}$-stable** if for any $E_0 \subseteq E$ separable there exists $E_1 \subseteq E$ separable such that $E_0 \subseteq E_1$ and E_1 is $\mathcal{Z}$-stable. In general we can replace "$\mathcal{Z}$-stable" with any other property we want.

Remark 4.1.9. Next, let's apply the classification of lifts to the trace-kernel extension. This fails immediately because B_ω is not separable nor $\mathcal{Z}$-stable (ultrapowers are not non-trivial tensor products for set-theoretic silliness). However, B_ω is separably $\mathcal{Z}$-stable when B is $\mathcal{Z}$-stable. Similarly, J_B is not stable, but we have a theorem telling us when J_B is separably stable.

Theorem 4.1.10. *When B is unital, simple, $\mathcal{Z}$-stable, $QT(B) = T(B)$ (for instance B is exact), then J_B is separably stable.*

Proof. (Sketch): We will assume B has real-rank zero. Then J_B does as well. Note that for $p \in J_B$ we have for any $\tau \in T(B_\omega)$ that $\tau(p) = 0 < 1 = \tau(1-p)$. By $\mathcal{Z}$-stability we get $p \precsim 1 - p$ so we get q as in the below theorem. In the non real-rank zero case, it's similar using positive elements and Cuntz comparison (and a slightly more technical condition), in place of the comparison by projections. □

Theorem 4.1.11 (Hjlemborg-Rørdam)**.** *For I real rank zero, I is separably stable iff for any projection $p \in \mathcal{P}(I)$, there exists a projection $q \in I$ such that $qp = 0$ and $p \sim q$ (Murray von Neumann equivalence).*

Computing the Invariant

Theorem 4.2.1 (Classifying Maps $A \to B^\omega$)**.** *Let A be separable unital nuclear, B unital $\mathcal{Z}$-stable. Then*

$$\{\textit{Unital } \theta : A \to B\}/\sim_u \;\cong\; \{\textit{Positive unital } \operatorname{Aff} T(A) \to \operatorname{Aff} T(B^\omega)\}$$
$$\theta \mapsto \operatorname{Aff} T(\theta)$$

Also, θ is full iff $\tau\theta \in T(A)$ is faithful for any $\tau \in T(B^\omega)$.

Definition 4.2.2. If A is separable, unital, and B is unital, $\mathcal{Z}$-stable, simple, $QT(B) = T(B)$, then for unital $\varphi, \psi : A \to B_\omega$ with $\operatorname{Aff} T(\varphi) = \operatorname{Aff} T(\psi)$, by the classification of lifts there is a unitary $u \in \mathcal{U}(B_\omega)$ with $\operatorname{Im}(\operatorname{Ad}(u)\varphi - \psi \subseteq J_B)$. We then define $\langle \varphi, \psi \rangle = [\operatorname{Ad}(u), \psi] \in KL(A, J_B)$.

Theorem 4.2.3. *$\langle \varphi, \psi \rangle$ is well-defined.*

Proof. (Sketch) For existence of u, we have

$$\begin{tikzcd} & & & A \arrow[d, bend right, "\psi"'] \arrow[d, bend left, "\varphi"] & \\ 0 \arrow[r] & J_B \arrow[r, "\iota"] & B_\omega \arrow[r, "q"] & B^\omega \arrow[r] & 0 \end{tikzcd}$$

And $\operatorname{Aff} T(\varphi) = \operatorname{Aff} T(\psi)$ implies $\operatorname{Aff} T(q\varphi) = \operatorname{Aff} T(q\psi)$, so there exists a unitary $\overline{u} \in B^\omega$ such that $\operatorname{Ad}(\overline{u})q\varphi = q\psi$ by the classification in Theorem 4.2.1. When B has unique trace, B^ω is a von Neumann algebra so $\mathcal{U}(B^\omega)$ is connected, hence $\overline{u}$ lifts to $u \in \mathcal{U}(B_\omega)$. In general (using $\mathcal{Z}$-stability) it is still true that $\mathcal{U}(B^\omega)$ is connected using "complemented partitions of unity" [Castillejos-Evington-Tikuisis-White-Winter in the nuclear case; and work of CCEGSTW for more general $\mathcal{Z}$-stable B]. So there exists $u \in \mathcal{U}(B_\omega)$ that lifts $\overline{u}$. Then $q\operatorname{Ad}(u)\varphi = \operatorname{Ad}(u)\psi$.

For $\langle \varphi, \psi \rangle$ being independent of the choice of u, the key idea is that $\mathcal{U}(B^\omega) \cap q(\varphi(A))'$ is path connected (again a consequence of complemented partitions of unity via $\mathcal{Z}$-stability). Then one uses ideas from the proof of the stable uniqueness theorem. □

This class $\langle \varphi \rangle \psi$ is the final piece of data needed to complete the classification of maps $A \to B_\infty$:

Theorem 4.2.4 (Classifying Maps $A \to B_\omega$)**.** *Suppose $\gamma : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B_\omega)$ is positive, unital, faithful (meaning $\gamma^* : T(B_\omega) \to T(A)$ has range in the faithful traces), and fix $\theta : A \to B^\omega$ full with $\operatorname{Aff} T(\theta) = \gamma$. Then*

1. *There exists $\psi : A \to B_\omega$ with $\operatorname{Aff} T(\theta) = \gamma$ iff there exists $\kappa \in KK(A, B_\omega)$ with $[q]\kappa = [\theta] \in KK(A, B^\omega)$ and $\kappa_0[1_A] = [1_B]$*

2. *Given $\psi : A \to B_\omega$ with $\operatorname{Aff} T(\psi) = \gamma$, we have*
$$\{\varphi : A \to B_\omega : \operatorname{Aff} T(\varphi) = \gamma\}/\sim_u \;\cong\; \{\kappa \in KL(A, J_B) : \kappa_0[1_A] = 0\}$$
via the map $\varphi \mapsto \langle \varphi, \psi \rangle$.

Without the UCT, to go further at present one needs a map from $A \to B$ to give a solution to the KK-lifting problem in part 1 of Theorem 4.2.4. If this map is assumed to induce an isomorphism at the level of KK and traces, then one can run an intertwining argument using the second part of the classification of maps $A \to B_\omega$ without using the UCT:

Theorem 4.2.5 (S)**.** *If A, B are USSN $\mathcal{Z}$-stable and $\varphi : A \to B$ is unital with $[\varphi] \in KK(A, B)$ and $T(\varphi) : T(B) \to T(A)$ are invertible, then $A \cong B$ via an isomorphism that is $\approx_u$ to φ.*

Another result without the UCT is that isomorphism of finite strongly self-absorbing algebras is determined by unit preserving KK-equivalence.

With the UCT, things simplify as the UCT can be used to calculate $KK(A, B_\omega$ and (via the universal multicoefficient theorem) $KL(A, J_B)$.

Proposition 4.2.6. *Let A, B be USSN $\mathcal{Z}$-stable UCT. There exists $\kappa \in KK(A, B_\omega)$ with $q\kappa = [\theta]$ and $\kappa_0[1] = [1]$ iff there exists $\alpha : K_0(A) \to K_0(B)$ with $K_0(q)\alpha = K_0(A)$ and $\alpha[1] = [1]$*

Proof. (Sketch) The key idea for the above proposition is that $\mathcal{U}_n(B^\omega)$ is connected for any $n \geq 1$, so $K_1(B^\omega) = 0$. And furthermore, projections in B^ω are classified by their trace(s), so $K_0(B^\omega) \cong \mathrm{Aff}\, T(B_\omega)$, Both of these results rely on the von Neumann like behaviour of B^ω which comes from $\mathcal{Z}$-stabiilty. Hence by the UCT $KK(A, B^\omega) \cong \mathrm{Hom}(K_0(A), K_0(B^\omega))$. Thus, given any $\alpha : K_0(A) \to K_0(B_\omega)$ with $K_0(q)\alpha = K_0(A)$ and $\alpha[1] = [1]$, any KK-lift of α to $\kappa \in KK(A, B_\omega)$ works (this is far from unique, but that does not matter). □

To illustrate these ideas, we show how these classification results pick up the quasidiagonality theorem of Tikuisis-White-Winter. The original proof of this theorem was one of the last components of the original proof of the stably finite classification theorem. In hindsight the lifting proof below, is a prototype for the abstract approach to classification.

Theorem 4.2.7 (Tikuisis-White-Winter)**.** *If A is separable, nuclear, UCT with a faithful trace, then A is QD.*

Proof. (Sketch) Let $\mathcal{Q} = \otimes_{n\geq 1} M_n$, then by Voiculescu A is QD iff $A \hookrightarrow \mathcal{Q}_\omega$ (this uses nuclearity of A). If $1 \in A$ and τ_A is a faithful trace then Connes' theorem gives the existence of an embedding $\theta : (A, \tau_A) \hookrightarrow R^\omega = \mathcal{Q}^\omega$. Thus we have

$$\begin{tikzcd}
 & & & A \arrow[dl, dashed, "?"'] \arrow[d, "\theta"] & \\
0 \arrow[r] & J_{\mathcal{Q}} \arrow[r, "\iota"'] & \mathcal{Q}_\omega \arrow[r, "q"'] & R^\omega \arrow[r] & 0
\end{tikzcd}$$

And using classification of lifts, it is enough to show there exists a K_0 lift preserving the unit

$$\begin{tikzcd}
 & K_0(A) \arrow[dl, dashed, "?"'] \arrow[d] & \\
K_0(\mathcal{Q}_\omega) \arrow[r, "q_*"'] & K_0(R^\omega) \arrow[r] & 0
\end{tikzcd}$$

The quotient map q_* is surjective since the rationals are dense in the reals. And $K_0(\mathcal{Q}_\omega), K_0(R^\omega)$ are vector spaces and q is linear, so such a lift does indeed exist. $\square$

Remark 4.2.8. Finally to compute $KL(A, J_B)$, the UMCT (which is equivalent to UCT) gives $KL(A, J_B) \cong \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)$. Then use the 6-term exact sequence in $\underline{K}$ applied to

$$0 \longrightarrow J_B \longrightarrow B_\omega \longrightarrow B^\omega \longrightarrow 0$$

This leads to the invariant $\underline{K}T_u(A) = (\underline{K}, \overline{K}_1^{\mathrm{alg}}, \mathrm{Aff}\, T, \rho)$ where we note that $K_1(J_B) = \overline{K}_1^{\mathrm{alg}}$. This leads to classifying embeddings $A \hookrightarrow B_\omega$ by $\underline{K}T_u$, which by intertwining leads to classification of maps $A \to B$. Lastly,

$$\overline{K}_1^{\mathrm{alg}} \cong K_1(A) \oplus \mathrm{Aff}\, T(A)/\overline{\mathrm{Im}(\rho : K_0(A) \to \mathrm{Aff}\, T(A))}$$

(non-canonically) and

$$K_*(A; \mathbb{Z}_n) \cong (K_*(A) \otimes \mathbb{Z}_n) \oplus \mathrm{Tor}(K_{*+1}(A), \mathbb{Z}_n)$$

also non-canonically. So, for the classification of algebras, any isomorphism $KT_u(A) \cong KT_u(B)$ lifts (non-canonically) to an isomorphism $\underline{K}T_u(A) \cong \underline{K}T_u(B)$. In this way we can rid of $\underline{K}$ and $\overline{K}_1^{\mathrm{alg}}$ and classify algebras by the usual Elliott invariant.

Day 5

Equivariant Classification

Remark 5.1.1. Throughout A, B are separable unital C*-algebras, and Γ is a countable discrete group, and $\alpha : \Gamma \curvearrowright A$ is an action unless otherwise specified. (For non unital algebras things work, but one has to take care about where the unitaries are).

Definition 5.1.2 (Conjugate)**.** We say $\alpha : \Gamma \curvearrowright A$ and $\beta : \Gamma \curvearrowright B$ are **conjugate** if there exists an equivariant isomorphism $A \to B$, ie. an isomorphism $\varphi : A \to B$ such that $\beta_g \varphi = \varphi \alpha_g$ for all $g \in G$.

Definition 5.1.3 (Outer Automorphism)**.** $\alpha : \Gamma \curvearrowright A$ is **outer** if for any $1 \neq g \in \Gamma$, we have $\alpha_g \notin \operatorname{Inn}(A) = \{\operatorname{Ad}(u) : u \in \mathcal{U}(A)\}$.

Theorem 5.1.4 (Connes 77', Jones 80')**.** *If Γ is finite, there exists a unique outer action $\Gamma \curvearrowright R$ up to conjugacy. In particular, all outer actions are conjugate to the Bernoulli shift $\Gamma \curvearrowright R^{\otimes \Gamma}$*

Remark 5.1.5. For infinite groups, conjugacy is often too strong. For equivariant maps $\varphi, \psi : (A, \alpha) \to (B, \beta)$ we have that $\varphi \approx_u \psi$ iff there exists unitaries $(u_n)_{n=1}^{\infty} \subseteq B$ such that $\|u_n \varphi(a) u_n^* - \psi(a)\| \to 0$ and $\|\beta_g(u_n) - u_n\| \to 0$ for all $a \in A, g \in G$. In finite groups we can always average and get lots of fixed points, but this is not always possible for infinite groups.

Also, note that if we have $\varphi : (A, \alpha) \to (B, \beta)$ and $\psi : (B, \beta) \to (A, \alpha)$ which are equivariant and $\psi\varphi \approx_u \mathrm{id}_A$ and $\varphi\psi \approx_u \mathrm{id}_B$, then we do not necessarily have (A, α) conjugate to (B, β). However this does imply (A, α) is cocycle conjugate to (B, β).

Definition 5.1.6 (Cocycle Conjugate)**.** We say $\alpha : \Gamma \curvearrowright A$ and $\beta : \Gamma \curvearrowright B$ are **cocycle conjugate** if there exists $\varphi : A \to B$ and $u : \Gamma \to \mathcal{U}(B)$ (just a function) such that $\operatorname{Ad}(u_g)\beta_g \varphi = \varphi \alpha_g$ for all $g \in G$, and $u_g \beta_g(u_h) = u_{gh}$. We write $(A, \alpha) \sim_{cc} (B, \beta)$. We call such a u a β-cocycle (or a β-1-cocycle).

Remark 5.1.7. If $(A, \alpha) \sim_{cc} (B, \beta)$ then $A \rtimes_\alpha \Gamma \cong B \rtimes_\beta \Gamma$.

Definition 5.1.8. Given $\beta : \Gamma \curvearrowright B$ and $u : \Gamma \to \mathcal{U}(B)$ is a β-cocycle, there is a natural action $\beta^u : \Gamma \curvearrowright B$ given by $\beta^u_g = \operatorname{Ad}(u_g)\beta_g$.

Remark 5.1.9. Note that β^u really is a group action. (This is one very good reason for working with cocycle conjugacy).

$$\begin{aligned}
\beta^u_g \beta^u_h &= \operatorname{Ad}(u_g)\beta_g \operatorname{Ad}(u_h)\beta_h \\
&= \operatorname{Ad}(u_g \beta_g(u_h))\beta_g \beta_h \\
&= \operatorname{Ad}(u_{gh})\beta_{gh} \\
&= \beta^u_{gh}
\end{aligned}$$

Theorem 5.1.10 (Oceanu 85')**.** *If Γ is a countable discrete amenable group there is a unique outer action $\Gamma \curvearrowright R$ up to cocycle conjugacy.*

Definition 5.1.11 (Cocycle Morphism)**.** A **cocycle morphism** $(\varphi, u) : (A, \alpha) \to (B, \beta)$ is a $*$-homomorphism $\varphi : A \to B$ along with a function $u : \Gamma \to \mathcal{U}(B)$ such that $\mathrm{Ad}(u_g)\beta_g\varphi = \varphi\alpha_g$ and $u_g\beta_g(u_h) = u_{gh}$. Composition is given by $(\psi, v) \circ (\varphi, u) = (\psi\varphi, \psi(u_\bullet)v_\bullet)$.

Example 5.1.12. For $\beta : \Gamma \curvearrowright \mathcal{U}(B)$ and $u \in \mathcal{U}(B)$ (so that $\mathrm{Ad}(u)$ is an automorphism), then we can define a cocycle morphism $\partial u : \Gamma \to \mathcal{U}(B)$ by $(\partial u)_g = u\beta_g(u)^*$. Then $(\mathrm{Ad}(u), \partial u) : (B, \beta) \to (B, \beta)$ is a cocycle morphism and these are the **inner** cocycle automorphisms.

Definition 5.1.13 (Approximately Unitarily Equivalent)**.** If we have $(\varphi, u), (\psi, v) : (A, \alpha) \to (B, \beta)$, then we write $(\varphi, u) \approx_u (\psi, v)$ if there exists unitaries $(w_n)_{n=1}^\infty \in \mathcal{U}(B)$ such that $(\mathrm{Ad}(w_n), \partial w_n) \circ (\varphi, u) \to (\psi, v)$ in point-norm, or equivalently,

$$\|w_n\varphi(a)w_n^* - \psi(a)\| \to 0$$

And

$$\|w_n u_g\beta_g(w_n)^* - v_g\| \to 0$$

for all $a \in A, g \in G$.

Another important reason for working with cocycle conjugacy is that the two-sided intertwining argument works in this framework. The abstract framework for this was developed by Szabó; previously a different style of intertwining argument was used (Kishimoto-Evans intertwining) for classification results for actions.

Theorem 5.1.14 (Szabó)**.** *Given cocycle morphisms $(\varphi, u) : (A, \alpha) \to (B, \beta)$ and $(\psi, v) : (B, \beta) \to (A, \alpha)$ such that $(\varphi, u) \circ (\psi, v) \approx_u \mathrm{id}_B$ and $(\psi, v) \circ (\varphi, u) \approx_u \mathrm{id}_A$, then $(A, \alpha) \sim_{cc} (B, \beta)$.*

Remark 5.1.15. For the above theorem to work you only need A, B separable, Γ countable, and this also works for twists by a 2-cocycle.

Theorem 5.1.16 (Gabe-Szabó)**.** *If A, B stable Kirchberg algebras, Γ amenable, $\alpha : \Gamma \curvearrowright A$ and $\beta : \Gamma \curvearrowright B$ outer actions, then $(A, \alpha) \sim_{cc} (B, \beta)$ iff $(A, \alpha) \sim_{KK^\Gamma} (B, \beta)$.*

Remark 5.1.17. There is a way to define KK^Γ by taking "Cuntz pairs" of cocycle maps $(A, \alpha) \underset{(\varphi,u)}{\overset{(\psi,v)}{\rightrightarrows}} (E, \gamma) \trianglerighteq (B \otimes K, \beta \otimes \mathrm{id})$.

Theorem 5.1.18 (Higson-Kasparov/Meyer-Nest)**.** *If Γ amenable torsion free, A, B separable (not necessarily unital), and $\kappa \in KK^\Gamma(A, B)$ with κ invertible in $KK(A, B)$, then κ is invertible in $KK^\Gamma(A, B)$.*

Remark 5.1.19. Take $A, B = \mathcal{O}_2$ so that $KK(A, B) = 0$. Then every element of $KK^\Gamma(\mathcal{O}_2, \mathcal{O}_2)$ is invertible (as $KK(\mathcal{O}_2, \mathcal{O}_2) = 0$ and so all elements are invertible there) hence every torsion-free amenable group has a unique outer action on $\mathcal{O}_2$ up to cocycle conjugacy.

In order to define the fundamental abstract property used by Gabe and Szabó we need a model action.

Definition 5.1.20. Let Γ be a countable infinite group. Write

$$\mathcal{O}_\Gamma = \left\{ s_g : g \in \Gamma : s_g^* s_h = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{else} \end{cases} \right\}$$

(this is just $\mathcal{O}_\infty$ but reindexed as Γ is infinite countable). We define $\gamma : \Gamma \curvearrowright \mathcal{O}_\Gamma$ by $\gamma_g(s_h) = s_{gh}$.

Definition 5.1.21 (Isometric Shift Absorbing)**.** An action $\beta : \Gamma \curvearrowright B$ is **isometric shift absorbing** (ISA) if there exists unital equivariant $(\mathcal{O}_\Gamma, \gamma) \hookrightarrow (B_\omega \cap B', \beta)$. This is equivalent to $(B, \beta) \otimes (\mathcal{O}_\Gamma, \gamma) \sim_{cc} (B, \beta)$.

Theorem 5.1.22. *If B is a unital Kirchberg algebra and $\beta : \Gamma \curvearrowright B$ an amenable outer action, then (B, β) is ISA*

Proof. (Sketch) Due to Kirchberg, $B_\omega \cap B'$ is simple purely infinite. Also, $\beta : \Gamma \to B_\omega \cap B'$ is outer. Then we can use a result of Kishimoto on the structure of outer actions on simple C*-algebras. □

Definition 5.1.23 ($\mathcal{O}_2$-Stable)**.** We say (A, α) is **$\mathcal{O}_2$-stable** if $(A, \alpha) \otimes (\mathcal{O}_2, \mathrm{id}_{\mathcal{O}_2}) \sim_{cc} (A, \alpha)$, or equivalently, if $\mathcal{O}_2 \hookrightarrow (A_\omega \cap A')^\alpha$ unitally where the superscript means the fixed points of α.

Theorem 5.1.24 (Gabe-Szabo)**.** *For Γ torsion-free amenable, all outer actions of $\Gamma \curvearrowright \mathcal{O}_2$ are $(\mathcal{O}_2, \mathrm{id}_{\mathcal{O}_2})$-stable.*

Sketch proof. Goal: For Γ amenable, $\alpha : \Gamma \curvearrowright \mathcal{O}_2, \beta : \Gamma \curvearrowright \mathcal{O}_2$ outer, then $\alpha \sim_{cc} \beta$. For convenience we will assume Γ is infinite.

This will be proved by establishing the following one-sided statement: For A unital separable nuclear, $\alpha : \Gamma \curvearrowright A$ and $\beta : \Gamma \curvearrowright \mathcal{O}_2$ outer, $\mathcal{O}_2$-stable, there exists

$$(\varphi, u) : (A, \alpha) \hookrightarrow (\mathcal{O}_2, \beta)$$

which is unique up to $\approx_u$.

For existence, we note that $A \rtimes_\alpha \Gamma$ is separable unital nuclear, so by Kirchberg's $\mathcal{O}_2$-embedding theorem $A \rtimes_\alpha \Gamma \hookrightarrow \mathcal{O}_2$. Set $\varphi = \widetilde{\varphi}|_A : A \to \mathcal{O}_2$ which is a $*$-morphism, and $u = \widetilde{\varphi}|_\Gamma : \Gamma \to \mathcal{U}(\mathcal{O}_2)$ which is a group homomorphism, and then $(\varphi, u) : (A, \alpha) \to (\mathcal{O}_2, \mathrm{id})$ is a cocycle morphism. Then consider

$$(A, \alpha) \xrightarrow{(\varphi, u)} (\mathcal{O}_2, \mathrm{id}) \xrightarrow{1 \otimes \mathrm{id}} (\mathcal{O}_2, \beta) \otimes (\mathcal{O}_2, \mathrm{id}) \to (\mathcal{O}_2, \beta)$$

The last map comes from $\mathcal{O}_2$-stability of β.

For uniqueness, For convenience we will assume that we have equivariant morphisms $\varphi, \psi : (A, \alpha) \to (\mathcal{O}_2, \beta)$ with A separable unital nuclear, and B separable $\mathcal{O}_2$-stable. We want to find a unitary $u \in \mathcal{U}(B)$ such that $\mathrm{Ad}(u)\varphi \approx \psi$ and $u \approx \beta_g(u)$. By non-equivariant classification, we can find $w_0 \in \mathcal{U}((\mathcal{O}_2)_\omega)$ such that $\mathrm{Ad}(w_0^*)\varphi = \psi$. Next, since $\mathcal{O}_2$ is ISA, we can find a sequence of orthogonal isometries $(s_g)_{g\in\Gamma} \subseteq (\mathcal{O}_2)_\omega \cap \mathcal{O}_2'$ with $\beta_g(s_h) = s_{gh}$. Fix finite $G \subseteq \Gamma$ and set

$$w = \frac{1}{|G|^{1/2}} \sum_{g\in G} s_g \beta_g(w_0) \in (\mathcal{O}_2)_\omega.$$

[12] Then we compute that (using commutativity and orthogonality of the isometries)

$$\begin{aligned}
w^*\varphi(a)w &= \frac{1}{|G|} \sum_{g,h\in G} \beta_g(w_0)^* s_g^* \varphi(a) s_h \beta_h(w_0) \\
&= \frac{1}{|G|} \sum_{g\in G} \beta_g(w_0^*)\varphi(a)\beta_g(w_0) \\
&= \frac{1}{|G|} \sum_{g\in G} \beta_g(w_0^*\varphi(\alpha_g^{-1}(a))w_0) \\
&= \frac{1}{|G|} \sum_{g\in G} \beta_g \psi(\alpha_g^{-1}(a)) \\
&= \frac{1}{|G|} \sum_{g\in G} \psi(a) \\
&= \psi(a)
\end{aligned}$$

We also note that w is an isometry and that

$$\begin{aligned}
\beta_g(w) &= \frac{1}{|G|^{1/2}} \sum_{h\in G} \beta_g(s_h \beta_h(w_0)) \\
&= \frac{1}{|G|^{1/2}} \sum_{h\in G} s_{gh} \beta_{gh}(w_0)
\end{aligned}$$

One can then check that

$$\|w - \beta_g(w)\|^2 = \frac{|gG\Delta G|}{|G|}$$

Then take a sequence of Følner sets. What we get is that there exists an isometry $w \in (\mathcal{O}_2)_\omega$ such that $w^*\varphi(a)w = \psi(a)$ and $\beta_g(w) = w$. We denote this by $\varphi \precsim_\Gamma \psi$. By symmetry, $\psi \precsim_\Gamma \varphi$.

[12] If we start with cocycle morphsisms (φ, u) and (ψ, v), take $w = \frac{1}{|G|^{1/2}} \sum_{g\in G} s_g u_g \beta_g(w) v_g^*$.

Finally, to from this subequivalence in both directions to obtain unitary equivalence, we use Connes' 2×2 matrix trick. Suppose we have $\varphi, \psi : A \to \mathcal{O}_2$ and consider $\pi = \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} : A \to M_2(\mathcal{O}_2)$. Then we work with $M_2(\mathcal{O}_2)_\omega \cap \pi(A)'$, where there are two canonical projections $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathcal{O}_2)_\omega \cap \pi(A)'$. If $p \sim q$ in $M_2(\mathcal{O}_2)_\omega \cap \pi(A)'$, then $\varphi \approx_u \psi$. This works compatibly with the actions: If $p \sim q$ in $(M_2(\mathcal{O}_2)_\omega \cap \pi(A)')^\Gamma$ then $\varphi \approx_u \psi$ equivariantly.

Returning to uniqueness, we have $\varphi \precsim_\Gamma \psi$ and $\psi \precsim_\Gamma \varphi$ which implies $p \precsim q$ and $q \precsim p$ in $(M_2(\mathcal{O}_2)_\omega \cap \pi(A)')^\Gamma$. And $p + q = 1$ implies p, q are full. A result of Cuntz states that if $p, q \in D$ are properly infinite full projections, then $p \sim q$ iff $[p]_0 = [q]_0 \in K_0(D)$. Now, using that β is $\mathcal{O}_2$-stable, we see that $(M_2(\mathcal{O}_2)_\omega \cap \pi(A)')^\Gamma$ is separably $\mathcal{O}_2$-stable. So p, q are properly infinite full, and $[p]_0 = [q]_0$ since $K_0((M_2(\mathcal{O}_2)_\omega \cap \pi(A)')^\Gamma) = 0$. Thus $p \sim q$ and so $\varphi \approx_u \psi$. $\square$