

A MILNOR EXACT SEQUENCE FOR E -THEORY

JOSÉ R. CARRIÓN

ABSTRACT. For separable C^* -algebras A and B , the E -theory group $E(A, B)$ carries a natural, generally non-Hausdorff, second-countable group topology, Hausdorff exactly when the closure of zero $\overline{\{0\}} \subseteq E(A, B)$ is trivial. The nonzero elements of $\overline{\{0\}}$ are the phantom classes, invisible to every continuous Hausdorff-valued invariant. We identify $\overline{\{0\}}$ as a derived inverse limit of homotopy classes of $*$ -homomorphisms over any shape system (C_n) of $SA \otimes \mathcal{K}$; $\overline{\{0\}} \cong \varprojlim^1 [C_n, S^2 B \otimes \mathcal{K}]$, giving a natural Milnor sequence

$$0 \rightarrow \varprojlim^1 [C_n, S^2 B \otimes \mathcal{K}] \rightarrow E(A, B) \rightarrow \varprojlim [C_n, SB \otimes \mathcal{K}] \rightarrow 0$$

for all separable A and B , with no UCT or nuclearity hypothesis. For nuclear A it agrees with the Willett–Yu controlled- KK Milnor sequence, and when A satisfies the UCT, nuclear or not, its $\varprojlim^1$ term is the pure-extension group $\text{Pext}_{\mathbb{Z}}^1(K_{*+1}(A), K_*(B))$, as in Schochet’s fine-structure description of the Kasparov groups.

1. INTRODUCTION

E -theory is a bivariant homology theory built from asymptotic morphisms. For separable A, B it is $E(A, B) = [[SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]]$, a group of asymptotic-homotopy classes of asymptotic morphisms, and it agrees with Kasparov’s $KK(A, B)$ when A is nuclear [CH90]. In [CS24] a natural second-countable group topology is placed on $E(A, B)$, the E -theoretic analog of Dadarlat’s topology on KK [Dad05], with which it agrees when A is nuclear: a sequence $x_m \rightarrow x$ if and only if there is $y \in E(A, C(\mathbb{N}^\dagger, B))$ with $y(m) = x_m$ and $y(\infty) = x$ (known as Pimsner’s condition), where $\mathbb{N}^\dagger = \mathbb{N} \cup \{\infty\}$. The topology is in general not Hausdorff. It is Hausdorff exactly when the closure of zero $\overline{\{0\}} \subseteq E(A, B)$ is trivial, and the Hausdorffized group $EL(A, B) := E(A, B)/\overline{\{0\}}$ is the E -theoretic counterpart of Rørdam’s KL [Rø95, Dad05]. For nuclear A one has $EL(A, B) = KL(A, B)$ [CS24, Section 4.2].

Rørdam introduced KL for A satisfying the UCT as the quotient of $KK(A, B)$ by the image of the pure extensions, because approximate unitary equivalence, central to the classification of nuclear C^* -algebras, detects a class only up to that subgroup [Rø95]. That image coincides with the closure of zero when A is nuclear and satisfies the UCT, and $KL(A, B) = KK(A, B)/\overline{\{0\}}$ became the general definition [Dad05]. Thus $\overline{\{0\}}$ is the information these approximate comparisons leave unresolved.

The nonzero elements of $\overline{\{0\}}$ are the *phantom* classes: E -theory classes that are not zero yet are identified with 0 by every continuous, Hausdorff-valued invariant. The name is borrowed from homotopy theory, where a phantom map is nontrivial but restricts to a null-homotopic map on every finite subcomplex, and where the phantom maps between two spaces are themselves identified with a $\varprojlim^1$ (cf. [Sch03,

2020 *Mathematics Subject Classification.* 19K35, 46L80, 46L85.

The author was partially supported by NSF grant DMS-2451675.

Section 12)). Theorem 1.1, the main result of this paper, is an E -theoretic analog of that description, and Corollary 4.2 makes the analogy precise: the phantom classes are exactly the classes that vanish on every stage of a shape system of $SA \otimes \mathcal{K}$.

In the KK -picture this subgroup has been described as a $\varprojlim^1$ in several settings. The first description of this kind goes back to work of L. Brown on Ext [Bro75, Bro77, Bro84]. When A satisfies the UCT, Schochet [Sch96, Sch98] constructed from a KK -filtration of A (a filtration that exists precisely when the UCT holds) a Milnor sequence for $KK(A, B)$ whose $\varprojlim^1$ term is the pure-extension group $\text{Pext}_{\mathbb{Z}}^1(K_{*+1}(A), K_*(B))$. The fine-structure papers [Sch01, Sch02] identify $\overline{\{0\}}$ with that term. Willett and Yu [WY26] identified $\overline{\{0\}}$ with a $\varprojlim^1$ of their controlled KK -groups.

We work in E -theory instead. For (K) -nuclear A one has $E(A, B) = KK(A, B)$ [CH90, Corollary 9], but in general the two theories differ, and it is E -theory that is half-exact in both variables for all separable C^* -algebras. Half-exactness and countable additivity already give E -theory a Milnor sequence of the classical kind: if A is presented as a direct limit $\varinjlim A_n$ of separable C^* -algebras, there is a natural exact sequence $0 \rightarrow \varprojlim^1 E(A_n, SB) \rightarrow E(A, B) \rightarrow \varprojlim E(A_n, B) \rightarrow 0$ [GHT00, Proposition 7.2], computing $E(A, B)$ from the chosen presentation (and making no reference to the topology on $E(A, B)$). Applied to a shape system (C_n) of $SA \otimes \mathcal{K}$, with second variable SB so that the middle term is again $E(A, B)$, it identifies its $\varprojlim^1$ term with the kernel of the restriction map $E(A, B) \rightarrow \varprojlim E(C_n, SB)$, the group of classes whose restrictions vanish in $E(C_n, SB)$. Bunke and Dünzinger have shown, as an application of a deep ∞ -categorical analysis of E -theory, that the $\varprojlim^1$ term of a sequence of this kind is the closure of zero: they prove that the E -theory category is compactly assembled and identify the $\varprojlim^1$ term of the resulting abstract Milnor sequence with $\overline{\{0\}}$ [BD24, Section 4]. The terms of that sequence are again E -theory groups of the stages: for (C'_n) a shape system of $S^2A \otimes \mathcal{K}$, they are the groups $E(C'_n, B) = [[SC'_n \otimes \mathcal{K}, SB \otimes \mathcal{K}]]$. In fact, we show that membership in $\overline{\{0\}}$ can be tested at a finer level: a class lies in $\overline{\{0\}}$ exactly when its restrictions vanish already as asymptotic-homotopy classes of $*$ -homomorphisms $C_n \rightarrow SB \otimes \mathcal{K}$, before suspension and stabilization (Corollary 4.2). Working directly with the homotopy classes over a shape system of $SA \otimes \mathcal{K}$, we prove that the identification holds at the level of these finer invariants, for arbitrary separable A and B , by an elementary argument in the language of asymptotic morphisms (Theorem 1.1). The resulting Milnor sequence is consistent with the descriptions above. For nuclear A , where $E(A, B) = KK(A, B)$, it and the Willett–Yu sequence describe the same extension of KL by $\overline{\{0\}}$ (Proposition 4.5). Whenever A satisfies the UCT, nuclear or not, its $\varprojlim^1$ term is a pure-extension group of the K -theory of A and B (Proposition 4.6).

Willett and Yu’s sequence has since proved fruitful: they used it to give a sufficient condition for the UCT [WY24]. The present description points the other way. For nuclear A , the $\varprojlim^1$ term of Theorem 1.1 computes $\overline{\{0\}}_{KK(A, B)}$ from a shape system alone, with no appeal to the universal coefficient theorem. A nuclear A whose $\varprojlim^1$ term is not isomorphic to $\text{Pext}_{\mathbb{Z}}^1(K_{*+1}(A), K_*(B))$, for even one B , would therefore be a counterexample to the UCT. We do not know whether the invariant can be made effective enough to exhibit such an algebra; we record only that it describes the obstruction without presupposing what it might detect in practice.

Recall [Bla85] that a *shape system* for a separable C^* -algebra C is an inductive system (C_n, γ_n) of separable C^* -algebras with semiprojective connecting maps and $\varinjlim C_n \cong C$ —every separable C^* -algebra admits one [Bla85, Theorem 4.3]. We write $[C, D]$ for the set of homotopy classes of $*$ -homomorphisms $C \rightarrow D$ (an abelian group when D is a suspension of a stable algebra; see Lemma 2.4). Our main result is the following.

Theorem 1.1. *Let A and B be separable C^* -algebras and let (C_n, γ_n) be a shape system for $SA \otimes \mathcal{K}$. Then the closure of zero in $E(A, B)$ is a first derived inverse limit,*

$$(1) \quad \overline{\{0\}} \cong \varprojlim^1 [C_n, S^2 B \otimes \mathcal{K}].$$

More precisely, there is a natural short exact sequence of abelian groups,

$$0 \longrightarrow \varprojlim^1 [C_n, S^2 B \otimes \mathcal{K}] \xrightarrow{\partial} E(A, B) \xrightarrow{\Theta} \varprojlim [C_n, SB \otimes \mathcal{K}] \longrightarrow 0,$$

in which Θ is the composition of the quotient $E(A, B) \rightarrow EL(A, B)$ with an isomorphism $EL(A, B) \cong \varprojlim [C_n, SB \otimes \mathcal{K}]$ (see Proposition 2.8), and the isomorphism (1) is implemented by ∂ , regarded as a map onto $\ker \Theta = \overline{\{0\}}$.

The identification made in the right-most term in the exact sequence above and the surjectivity of Θ are essentially the Hausdorffization theorem of [CS24] (recalled here as Theorem 2.2(iv)); see Lemma 2.6. The content added here is the connecting map ∂ defined on a $\varprojlim^1$ of homotopy classes of $*$ -homomorphisms, and the identification of the closure of zero with that group.

The proof works entirely at the level of asymptotic morphisms, using Dadarlat's homotopy limit [Dad94] and the Pimsner condition of [CS24] in place of the homotopy theory of the mapping spaces $\text{Hom}(C_n, SB \otimes \mathcal{K})$: with the point-norm topology the restriction maps between these spaces need not be fibrations, so the classical Milnor sequence for π_0 of the limit of a tower of fibrations is not directly available.

Theorem 1.1 is a special case of a statement about the bifunctor $[[\cdot, \cdot]]$ on asymptotic morphisms (Theorem 3.1). E -theory is recovered by taking source $SA \otimes \mathcal{K}$ and target $SB \otimes \mathcal{K}$.

Section 2 recalls the framework of [CS24] and proves the comparison of towers used to pass between homotopy and asymptotic-homotopy classes. Section 3 constructs the connecting map and proves Theorems 1.1 and 3.1. Section 4 treats naturality, a Hausdorffness criterion, the dichotomy, and the comparison with the sequences of Willett–Yu and Schochet.

2. PRELIMINARIES

We recall the framework of [CS24]. We refer the reader to [CH90, Bla98] for more information on asymptotic morphisms and E -theory and to [Bla85, Dad94] for semiprojectivity, shape systems, and the homotopy limit.

Throughout, C and D are C^* -algebras, with C separable whenever it is the source of an asymptotic morphism. An *asymptotic morphism* $\phi: C \xrightarrow{\sim} D$ is a family of self-adjoint linear maps $(\phi_t)_{t \geq 0}$, point-norm continuous in t , with $\|\phi_t(xy) - \phi_t(x)\phi_t(y)\| \rightarrow 0$ for all $x, y \in C$. Two asymptotic morphisms $\phi, \psi: C \xrightarrow{\sim} D$ are *equivalent* if $\|\phi_t(x) - \psi_t(x)\| \rightarrow 0$ for all $x \in C$, and *asymptotically homotopic* if there is an asymptotic morphism $C \xrightarrow{\sim} C([0, 1], D)$ whose evaluations at 0 and 1 are equivalent to ϕ and ψ . We write $[[C, D]]$ for the set of asymptotic-homotopy classes

and, following [CS24], $H(C, D) \subseteq [[C, D]]$ for the image of $\text{Hom}(C, D)$ in $[[C, D]]$: genuine $*$ -homomorphisms modulo asymptotic homotopy, carrying the quotient topology from the point-norm topology on $\text{Hom}(C, D)$. There is no reason for the natural surjection $[C, D] \rightarrow H(C, D)$ from genuine homotopy classes to be injective. We compare the two more systematically in Lemma 2.6. A $*$ -homomorphism ρ with domain D acts by post-composition, $\rho_*[[\phi]] = [[\rho \circ \phi]]$, and a $*$ -homomorphism σ with codomain C by pre-composition, $\sigma^*[[\phi]] = [[\phi \circ \sigma]]$. More generally, asymptotic morphisms compose, up to asymptotic homotopy, making the sets $[[C, D]]$ the morphism sets of a category; see [CH90] or [Bla98, Section 25.3]. A *reparametrization* of ϕ is $(\phi_{r(t)})_t$ for a continuous $r: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $r(t) \rightarrow \infty$. Every asymptotic morphism is asymptotically homotopic to each of its reparametrizations [Bla98, 25.1.2(h)]. If $\sigma: C_0 \rightarrow C$ is a *semiprojective* $*$ -homomorphism [Bla85, Definition 2.10] with C_0 separable, then $\phi \circ \sigma$ is asymptotically homotopic to a genuine $*$ -homomorphism for every asymptotic morphism $\phi: C \xrightarrow{\sim} D$, so that σ^* maps $[[C, D]]$ into $H(C_0, D)$ [CS24, Lemma 2.2].

Definition 2.1 ([CS24, Definition 2.3]). For C separable, $[[C, D]]$ carries the weakest topology making $\sigma^*: [[C, D]] \rightarrow H(C_0, D)$ continuous for every separable C_0 and every semiprojective $*$ -homomorphism $\sigma: C_0 \rightarrow C$.

Fix a shape system (C_n, γ_n) for C , with canonical maps $\gamma_{\infty, n}: C_n \rightarrow C$, and write $\gamma_{m, n} := \gamma_{m-1} \circ \dots \circ \gamma_n: C_n \rightarrow C_m$ for the composite connecting maps ($m \geq n$, with $\gamma_{n, n} = \text{id}$). Precomposition with γ_n gives structure maps γ_n^* on each of the towers $([C_n, D])_n$ and $(H(C_n, D))_n$.

The following combines Theorem 2.10, Proposition 2.8, Theorem 2.12, Theorem 3.7, and Lemma 2.7 of [CS24].

Theorem 2.2. *Let C be separable with shape system (C_n, γ_n) and let D be a C^* -algebra.*

- (i) $[[C, D]]$ is first countable; second countable if D is separable.
- (ii) For each n , the map $\gamma_n^*: H(C_{n+1}, D) \rightarrow H(C_n, D)$ factors through a discrete space that is countable when D is separable.
- (iii) (Pimsner's condition.) For $x_m, x \in [[C, D]]$, $x_m \rightarrow x$ if and only if there is $y \in [[C, C(\mathbb{N}^\dagger, D)]]$ with $y(m) = x_m$ ($m \in \mathbb{N}$) and $y(\infty) = x$.
- (iv) (Hausdorffization.) The maps $\gamma_{\infty, n}^*$ induce a homeomorphism $[[C, D]]_{\text{Hd}} \cong \varprojlim (H(C_n, D), \gamma_n^*)$, where $[[C, D]]_{\text{Hd}}$ is the Kolmogorov (i.e., T_0 , but in this case, Hausdorff) quotient.
- (v) If $\alpha: C_0 \rightarrow C_1$ is a semiprojective $*$ -homomorphism between separable C^* -algebras and $\phi, \psi: C_1 \rightarrow D$ are $*$ -homomorphisms with $[[\phi]] = [[\psi]]$ in $[[C_1, D]]$, then $\phi \circ \alpha$ and $\psi \circ \alpha$ are homotopic through $*$ -homomorphisms.

A *strong homotopy morphism* $(\underline{\phi}, \underline{h}): (C_n, \gamma_n) \rightarrow D$ (the case of Dadarlat's *strong maps of systems* [Dad94, Definition 1.5] in which the target system is the constant system D) consists of $*$ -homomorphisms $\phi_n: C_n \rightarrow D$ and homotopies $h_n: C_n \rightarrow C([0, 1], D)$ with $\text{ev}_0 h_n = \phi_n$ and $\text{ev}_1 h_n = \phi_{n+1} \circ \gamma_n$. Its *homotopy limit* $\text{h-}\varprojlim(\underline{\phi}, \underline{h})$, an asymptotic morphism $C \xrightarrow{\sim} D$, and the accompanying notion of homotopy between strong homotopy morphisms are due to Dadarlat [Dad94, Section 2 and Definition 1.5]. The homotopy limit is recalled in detail in [CS24, Section 1]. Briefly, $\text{h-}\varprojlim(\underline{\phi}, \underline{h})$ concatenates the homotopies over unit intervals: for

$a \in C_n$ and $t \in [m, m+1]$ with $m \geq n$,

$$(2) \quad \text{h-}\varinjlim (\phi, \underline{h})_t(a) = h_m(\gamma_{m,n}(a))(t-m),$$

and this determines the asymptotic morphism up to equivalence [Dad94, Section 2]. A strong homotopy morphism is precisely a point of the homotopy inverse limit of the tower of mapping spaces $(\text{Hom}(C_n, D))_n$ with the point-norm topology: a point of each space together with a path from it to the image of the next (cf. [BK72]). Part (ii) of the following theorem says that $[[C, D]]$ is π_0 of this homotopy limit.

Theorem 2.3 (Dadarlat). *Let C be separable with shape system (C_n, γ_n) and let D be a separable C^* -algebra.*

- (i) *Every asymptotic morphism $C \xrightarrow{\sim} D$ is asymptotically homotopic to $\text{h-}\varinjlim (\phi, \underline{h})$ for some strong homotopy morphism $(\phi, \underline{h}): (C_n, \gamma_n) \rightarrow D$ [Dad94, Corollary 3.15]. In fact, $[[\text{h-}\varinjlim (\phi, \underline{h}) \circ \gamma_{\infty, n}]] = [[\phi_n]]$ in $\text{H}(C_n, D)$ for all n [CS24, Proposition 1.5].*
- (ii) *The homotopy limit induces a bijection from homotopy classes of strong homotopy morphisms $(C_n, \gamma_n) \rightarrow D$ onto $[[C, D]]$ [Dad94, Theorem 3.5]. In particular, two strong homotopy morphisms with asymptotically homotopic homotopy limits are homotopic [Dad94, Proposition 3.18].*

We will often identify continuous maps of compact metric spaces into mapping spaces with $*$ -homomorphisms: for a compact metric space X , $*$ -homomorphisms $C \rightarrow C(X, D)$ correspond to point-norm continuous maps $X \rightarrow \text{Hom}(C, D)$. In particular, writing $SD = C_0((0, 1), D)$, a $*$ -homomorphism $C \rightarrow SD$ is a based loop at 0 in $\text{Hom}(C, D)$, a $*$ -homomorphism $C \rightarrow C([0, 1], SD)$ is a based homotopy of such loops, and a $*$ -homomorphism $C \rightarrow C([0, 1]^2, D)$ is a square in $\text{Hom}(C, D)$ with prescribed boundary.

When D is stable there are isometries v_1, v_2 in the multiplier algebra of D with $v_1 v_1^* + v_2 v_2^* = 1$, and the *orthogonal sum* of $\phi, \psi \in \text{Hom}(C, D)$ is $\phi \oplus \psi := v_1 \phi(\cdot) v_1^* + v_2 \psi(\cdot) v_2^*$. Up to homotopy it does not depend on the choice of v_1, v_2 .

Lemma 2.4 (cf. [Bla98, Proposition 25.4.3]). *Let C be separable and let D be a stable C^* -algebra. Then $[C, SD] = \pi_1(\text{Hom}(C, D), 0)$, the loop-concatenation sum agrees with the orthogonal sum, and $[C, SD]$ is an abelian group. Concatenation in the suspension coordinate likewise agrees with the orthogonal sum on $[[C, SD]]$, which is an abelian group as well.*

Proof. The identification $[C, SD] = \pi_1(\text{Hom}(C, D), 0)$ is a special case of the correspondence above. Both the loop sum $*$ and the orthogonal sum $\oplus$ are unital binary operations on $[C, SD]$ with common unit 0, and they satisfy the interchange law $(a \oplus b) * (c \oplus d) = (a * c) \oplus (b * d)$: at suspension parameter $s \leq \frac{1}{2}$ both sides are $v_1 a(2s) v_1^* + v_2 b(2s) v_2^*$, and at $s \geq \frac{1}{2}$ both are $v_1 c(2s-1) v_1^* + v_2 d(2s-1) v_2^*$. Moreover $a \oplus 0 \simeq a$ and $0 \oplus b \simeq b$ through loops. To see this, choose a strictly continuous path of isometries from 1 to v_i in the multiplier algebra, which exists since D is stable (cf. [Bla98, 25.4.1]). Conjugating along it gives such a homotopy. Hence

$$a * b = (a \oplus 0) * (0 \oplus b) = (a * 0) \oplus (0 * b) = a \oplus b,$$

so the two sums coincide. Commutativity follows in the same way, and inverses are given by loop reversal. The statements for asymptotic morphisms are [Bla98, 25.4.1 and Proposition 25.4.3]. $\square$

Recall that a *tower* of abelian groups is an inverse sequence $G_1 \xleftarrow{g_1} G_2 \xleftarrow{g_2} \dots$ with structure maps $g_n: G_{n+1} \rightarrow G_n$, and that, writing $\text{sh}((x_n)_n) = (g_n(x_{n+1}))_n$ for the shift on $\prod_n G_n$,

$$\varprojlim G_n = \ker(1 - \text{sh}) \quad \text{and} \quad \varprojlim^1 G_n = \text{coker}(1 - \text{sh});$$

see e.g. [Sch03, Section 3]. The next lemma, which compares towers that interleave, is folklore—we include the short proof since we could not locate a reference.

Lemma 2.5. *Let (G_n, g_n) and (K_n, k_n) be towers of abelian groups and suppose there are homomorphisms $q_n: G_n \rightarrow K_n$ and $r_n: K_{n+1} \rightarrow G_n$ with*

$$r_n \circ q_{n+1} = g_n \quad \text{and} \quad q_n \circ r_n = k_n$$

for all n . Then $(q_n)_n$ and $(r_n)_n$ induce mutually inverse isomorphisms $\varprojlim G_n \cong \varprojlim K_n$ and $\varprojlim^1 G_n \cong \varprojlim^1 K_n$.

Proof. The hypotheses say precisely that the alternating tower

$$G_1 \xleftarrow{r_1} K_2 \xleftarrow{q_2} G_2 \xleftarrow{r_2} K_3 \xleftarrow{q_3} G_3 \xleftarrow{r_3} \dots$$

contains both (G_n, g_n) and (K_{n+1}, k_{n+1}) as cofinal subsequences: consecutive maps compose to $r_n \circ q_{n+1} = g_n$ and $q_{n+1} \circ r_{n+1} = k_{n+1}$. Since $\varprojlim$ and $\varprojlim^1$ are unchanged by passage to cofinal subsequences [Sch03, Proposition 3.3], both towers compute the $\varprojlim$ and $\varprojlim^1$ of the alternating tower, via the maps induced by $(q_n)_n$ and $(r_n)_n$. $\square$

The isomorphism of Theorem 2.2(iv) is stated in terms of $H(C_n, D)$, the $*$ -homomorphisms modulo asymptotic homotopy, yet the constructions of Section 3 take place in the genuine homotopy classes $[C_n, D]$. The two need not agree stage by stage (asymptotically homotopic $*$ -homomorphisms out of C_n need not be homotopic, since C_n itself is not assumed semiprojective), but the next lemma shows that the comparison of consecutive stages furnished by Theorem 2.2(v) makes them agree after passing to limits, which is all that the inverse limit and its first derived functor see.

Lemma 2.6. *Let (C_n, γ_n) be a shape system for the separable algebra C , and let D be the suspension of a stable C^* -algebra, so that $[C_n, D]$ and its quotient $H(C_n, D)$ are abelian groups (Lemma 2.4). The natural surjections $q_n: [C_n, D] \rightarrow H(C_n, D)$ and the maps*

$$r_n: H(C_{n+1}, D) \longrightarrow [C_n, D], \quad [[\phi]] \longmapsto [\phi \circ \gamma_n],$$

satisfy the hypotheses of Lemma 2.5 for the towers with structure maps γ_n^ . Consequently,*

$$\varprojlim [C_n, D] \cong \varprojlim H(C_n, D) \quad \text{and} \quad \varprojlim^1 [C_n, D] \cong \varprojlim^1 H(C_n, D).$$

Proof. If $[[\phi]] = [[\psi]]$ in $H(C_{n+1}, D)$ then $\phi \circ \gamma_n$ and $\psi \circ \gamma_n$ are homotopic through $*$ -homomorphisms by Theorem 2.2(v), each γ_n being semiprojective. Therefore, r_n is well-defined, and it is a homomorphism because pre-composition is compatible with the orthogonal sum, as is the quotient map q_n . For a $*$ -homomorphism $\phi: C_{n+1} \rightarrow D$ we have $r_n(q_{n+1}[\phi]) = [\phi \circ \gamma_n] = \gamma_n^*[\phi]$ and $q_n(r_n[[\phi]]) = [[\phi \circ \gamma_n]] = \gamma_n^*[[\phi]]$, which are the hypotheses of Lemma 2.5. $\square$

To also identify $[[C, D]]_{\text{Hd}}$ with the quotient by the closure of zero, we record that $[[C, D]]$ is a topological group in the generality we need—the argument of [CS24, Theorem 4.5] for $E(A, B)$ applies unchanged.

Lemma 2.7. *Let C be separable and let D be the suspension of a stable C^* -algebra. Then $[[C, D]]$ is a topological abelian group under the orthogonal sum, and its Kolmogorov quotient $[[C, D]]_{\text{Hd}}$ is the quotient $[[C, D]]/\overline{\{0\}}$ by the closure of zero. In particular $[[C, D]]_{\text{Hd}}$ is Hausdorff.*

Proof. $[[C, D]]$ is an abelian group by Lemma 2.4. For continuity of subtraction, suppose $x_m \rightarrow x$ and $y_m \rightarrow y$. Pimsner's condition (Theorem 2.2(iii)) provides $\hat{x}, \hat{y} \in [[C, C(\mathbb{N}^\dagger, D)]]$ with $\hat{x}(m) = x_m$, $\hat{y}(m) = y_m$ for $m \in \mathbb{N}$ and $\hat{x}(\infty) = x$, $\hat{y}(\infty) = y$. Since $C(\mathbb{N}^\dagger, D)$ is again the suspension of a stable C^* -algebra, the difference $\hat{x} - \hat{y}$ is defined, and $(\hat{x} - \hat{y})(m) = x_m - y_m$ for $m \in \mathbb{N}$ and $(\hat{x} - \hat{y})(\infty) = x - y$, the evaluations being suspensions of $*$ -homomorphisms and hence compatible with concatenation in the suspension coordinate, which computes the sum (Lemma 2.4). Thus $x_m - y_m \rightarrow x - y$ by Theorem 2.2(iii) again. First countability (Theorem 2.2(i)) upgrades this to continuity of subtraction, so $[[C, D]]$ is a topological group. Translations are then homeomorphisms, so two classes are topologically indistinguishable exactly when their difference lies in $\overline{\{0\}}$. Hence the Kolmogorov quotient is the quotient by $\overline{\{0\}}$, and it is Hausdorff because the T_0 quotient of a topological group is Hausdorff. $\square$

Proposition 2.8. *Let C be separable with shape system (C_n, γ_n) and let D be the suspension of a stable C^* -algebra. There is a surjective homomorphism*

$$(3) \quad \Theta: [[C, D]] \longrightarrow \varprojlim [C_n, D] \quad \text{with} \quad \ker \Theta = \overline{\{0\}},$$

namely the composition of the quotient map onto $[[C, D]]_{\text{Hd}} = [[C, D]]/\overline{\{0\}}$ with the isomorphisms $[[C, D]]_{\text{Hd}} \cong \varprojlim H(C_n, D) \cong \varprojlim [C_n, D]$.

Proof. Lemma 2.7 identifies the Kolmogorov quotient $[[C, D]]_{\text{Hd}}$ with $[[C, D]]/\overline{\{0\}}$. The homeomorphism $[[C, D]]_{\text{Hd}} \cong \varprojlim H(C_n, D)$ of Theorem 2.2(iv) is additive, each $\gamma_{\infty, n}^*$ being additive, hence an isomorphism of groups, and Lemma 2.6 provides the isomorphism $\varprojlim H(C_n, D) \cong \varprojlim [C_n, D]$. $\square$

E -theory and its Hausdorffization are the instances $E(A, B) = [[SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]]$ and $EL(A, B) = E(A, B)/\overline{\{0\}}$ [CS24, Section 4.2], where Equation (3) reads

$$EL(A, B) \cong \varprojlim [C_n, SB \otimes \mathcal{K}]$$

and yields the surjection Θ and the right-hand term of Theorem 1.1.

3. THE CONNECTING MAP AND THE MILNOR SEQUENCE

Throughout this section C is separable with a fixed shape system (C_n, γ_n) , and D is the suspension of a separable stable C^* -algebra, hence itself stable, so that $[[C, D]]$, $[C_n, D]$, and $[C_n, SD]$ are abelian groups with the class of the zero $*$ -homomorphism as neutral element (Lemma 2.4). Theorem 1.1 will follow from the following more general statement (it is the case $C = SA \otimes \mathcal{K}$, $D = SB \otimes \mathcal{K}$).

Theorem 3.1. *Let C be a separable C^* -algebra with shape system (C_n, γ_n) and let D be the suspension of a separable stable C^* -algebra. With Θ as in Proposition 2.8, there is an exact sequence of abelian groups*

$$0 \rightarrow \varprojlim^1 [C_n, SD] \xrightarrow{\partial} [[C, D]] \xrightarrow{\Theta} \varprojlim [C_n, D] \rightarrow 0.$$

The rest of this section is devoted to its proof.

3.1. The connecting homomorphism. We consider the tower $([C_n, SD])_n$ with structure maps γ_n^* and, as in Section 2, realize $\varprojlim^1 [C_n, SD]$ as the cokernel of $1 - \text{sh}$ on $\prod_n [C_n, SD]$, where now $\text{sh}((\eta_n)_n) = (\gamma_n^*(\eta_{n+1}))_n$.

Let $\eta_n: C_n \rightarrow SD$ ($n \in \mathbb{N}$) be $*$ -homomorphisms, i.e., loops at 0 in $\text{Hom}(C_n, D)$. The pair $(\underline{0}, \underline{\eta})$ is then a strong homotopy morphism $(C_n, \gamma_n) \rightarrow D$ whose $*$ -homomorphisms are all zero. Indeed, the homotopy η_n is required to run from the zero map to the zero map, which is exactly what being a loop at 0 says. Define

$$\partial_0: \prod_n [C_n, SD] \longrightarrow [[C, D]], \quad \partial_0(([\eta_n])_n) := [[\text{h-}\varprojlim(\underline{0}, \underline{\eta})]];$$

see Figure 1. This does not depend on the choice of representatives. For this, note that if $\eta_n \simeq \eta'_n$ through loops for every n , then these homotopies, squares in $\text{Hom}(C_n, D)$ whose other two edges are constant at 0, are exactly the data of a homotopy of strong homotopy morphisms from $(\underline{0}, \underline{\eta})$ to $(\underline{0}, \underline{\eta}')$, so the two homotopy limits agree in $[[C, D]]$ by Theorem 2.3(ii).

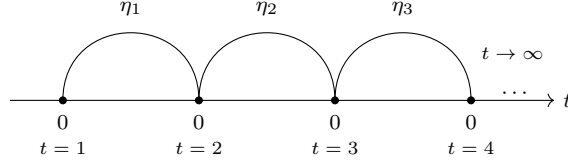


FIGURE 1. The connecting map ∂ : the loop η_n runs over $t \in [n, n+1]$, returning to 0 at each integer, so $\partial_0(\eta)$ vanishes on every stage C_n yet can represent a nonzero class of $[[C, D]]$.

Lemma 3.2. $\partial_0: \prod_n [C_n, SD] \rightarrow [[C, D]]$ is a group homomorphism, and $\Theta \circ \partial_0 = 0$.

Proof. Additivity: the orthogonal sum of two homotopy limits is the homotopy limit of the levelwise orthogonal sum of the data (immediate from Equation (2)), and choosing the orthogonal sums $\eta_n \oplus \eta'_n$ as representatives of $(\eta + \eta')_n$ (Lemma 2.4) makes the homotopy data of $\partial_0(\eta) + \partial_0(\eta')$ and of $\partial_0(\eta + \eta')$ coincide. For $\Theta \circ \partial_0 = 0$: by Theorem 2.3(i), $[[\text{h-}\varprojlim(\underline{0}, \underline{\eta}) \circ \gamma_{\infty, n}]] = [[0]] = 0$ in $H(C_n, D)$ for all n , and by its construction in Proposition 2.8 the element $\Theta(\partial_0(\eta))$ is computed from these restrictions. Therefore, it vanishes. $\square$

The key point is the following shift-invariance, where the reparametrization-invariance of asymptotic morphisms enters.

Lemma 3.3. $\partial_0 \circ \text{sh} = \partial_0$. Consequently, $\partial_0 \circ (1 - \text{sh}) = 0$ and ∂_0 descends to a homomorphism

$$\partial: \varprojlim^1 [C_n, SD] \longrightarrow \overline{\{0\}} \subseteq [[C, D]].$$

Proof. Fix $*$ -homomorphisms $\zeta_n: C_n \rightarrow SD$ ($n \in \mathbb{N}$), write $\zeta = (\zeta_n)_n$, and set $\psi = \text{h-}\varprojlim(\underline{0}, \underline{\zeta})$, realized via Equation (2): for $a \in C_n$ and $t \in [m, m+1]$ ($m \geq n$), $\psi_t(a) = \zeta_m(\gamma_{m,n}(a))(t-m)$, where $\zeta_m(\cdot)(s) \in D$ denotes the loop ζ_m at parameter $s \in [0, 1]$ (vanishing at $s = 0, 1$). The shifted sequence is $(\text{sh } \zeta)_m = \gamma_m^* \zeta_{m+1} = \zeta_{m+1} \circ \gamma_m$, so for $a \in C_n$ and $t \in [m, m+1]$,

$$\begin{aligned} (\text{h-}\varprojlim(\underline{0}, \underline{\text{sh } \zeta}))_t(a) &= \zeta_{m+1}(\gamma_m(\gamma_{m,n}(a)))(t-m) \\ &= \zeta_{m+1}(\gamma_{m+1,n}(a))(t-m) = \psi_{t+1}(a), \end{aligned}$$

using $\gamma_m \circ \gamma_{m,n} = \gamma_{m+1,n}$ and the same formula for ψ on $[m+1, m+2]$. Thus $\text{h-}\varinjlim(0, \text{sh } \zeta)$ is the reparametrization $t \mapsto t+1$ of ψ , hence asymptotically homotopic to ψ [Bla98, 25.1.2(h)]. Hence $\partial_0(\text{sh } \zeta) = [[\psi]] = \partial_0(\zeta)$.

Since ∂_0 is a homomorphism (Lemma 3.2), $\partial_0 \circ (1 - \text{sh}) = \partial_0 - \partial_0 \circ \text{sh} = 0$. As $\varinjlim^1[C_n, SD]$ is the cokernel of $1 - \text{sh}$, ∂_0 descends to ∂ . Finally, $\text{im } \partial \subseteq \overline{\{0\}}$ because $\Theta \circ \partial_0 = 0$ (Lemma 3.2). $\square$

3.2. Exactness. Now we show that ∂ is injective and that its image is all of $\overline{\{0\}}$. Surjectivity comes first: from the homotopy data of a representation of a class in $\overline{\{0\}}$ as a homotopy limit, we build an explicit preimage under ∂ .

Lemma 3.4. *Let $\psi = \text{h-}\varinjlim(\phi, h)$ be the homotopy limit of a strong homotopy morphism $(\phi, h): (C_n, \gamma_n) \rightarrow D$ and suppose $[[\psi]] \in \overline{\{0\}}$. Then each ϕ_n is null-homotopic through $*$ -homomorphisms, and for any null-homotopies $k_n: C_n \rightarrow C([0, 1], D)$ with $\text{ev}_0 k_n = \phi_n$ and $\text{ev}_1 k_n = 0$, the concatenations*

$$(4) \quad \Lambda_n := \overline{k_n} * h_n * (k_{n+1} \circ \gamma_n)$$

are loops at 0, and $\partial_0(([\Lambda_n])_n) = [[\psi]]$. In particular ∂ maps $\varinjlim^1[C_n, SD]$ onto $\overline{\{0\}}$.

Proof. Since $[[\psi]] \in \overline{\{0\}} = \ker \Theta$, the construction of Θ in Proposition 2.8 gives $[[\phi_n]] = [[\psi \circ \gamma_{\infty, n}]] = 0$ in $H(C_n, D)$ for all n , using Theorem 2.3(i) for the first equality. This records only that each ϕ_n is asymptotically null-homotopic, but in fact ϕ_n is null-homotopic through $*$ -homomorphisms. This is because γ_n is semiprojective and $[[\phi_{n+1}]] = 0$, so Theorem 2.2(v) provides a homotopy through $*$ -homomorphisms from $\phi_{n+1} \circ \gamma_n$ to 0, and concatenating the homotopy h_n (from ϕ_n to $\phi_{n+1} \circ \gamma_n$) with it gives one from ϕ_n to 0. Fix null-homotopies k_n as in the statement. In Equation (4), $\overline{k_n}$ runs from $\text{ev}_1 k_n = 0$ to $\text{ev}_0 k_n = \phi_n$; then h_n from ϕ_n to $\phi_{n+1} \circ \gamma_n$; then $k_{n+1} \circ \gamma_n$ from $\text{ev}_0(k_{n+1} \circ \gamma_n) = \phi_{n+1} \circ \gamma_n$ to 0. Thus Λ_n is a loop at 0, i.e., a $*$ -homomorphism $C_n \rightarrow SD$.

Now build a strong homotopy morphism $(\Phi, H): (C_n) \rightarrow C([0, 1], D)$ with $\Phi_n := k_n$ (the path $\phi_n \rightsquigarrow 0$) and H_n a filling of the square

$$\begin{array}{ccc} 0 & \xrightarrow{\Lambda_n} & 0 \\ k_n \uparrow & H_n & \uparrow k_{n+1} \circ \gamma_n \\ \phi_n & \xrightarrow{h_n} & \phi_{n+1} \circ \gamma_n \end{array}$$

whose horizontal edges (h_n below, Λ_n above) run along the homotopy parameter of H_n , and whose vertical edges (k_n and $k_{n+1} \circ \gamma_n$) along the $[0, 1]$ -coordinate of the target $C([0, 1], D)$. The filling exists because the two ways around the boundary are homotopic with endpoints fixed, since $k_n * \overline{k_n}$ cancels:

$$k_n * \Lambda_n = k_n * \overline{k_n} * h_n * (k_{n+1} \circ \gamma_n) \simeq h_n * (k_{n+1} \circ \gamma_n),$$

and such a homotopy, suitably reparametrized, is exactly such a filling. Its ends are $\text{ev}_0(\Phi, H) = (\phi, h)$ and $\text{ev}_1(\Phi, H) = (0, \underline{\Lambda})$, so by functoriality of $\text{h-}\varinjlim$ under post-composition with the evaluations $\text{ev}_t: C([0, 1], D) \rightarrow D$ the asymptotic morphism $\text{h-}\varinjlim(\Phi, H)$ is a homotopy from $\psi = \text{h-}\varinjlim(\phi, h)$ to $\text{h-}\varinjlim(0, \underline{\Lambda})$. Hence $\partial_0(([\Lambda_n])_n) = [[\text{h-}\varinjlim(0, \underline{\Lambda})]] = [[\psi]]$. Finally, every class in $\overline{\{0\}}$ is the class of such a homotopy limit (Theorem 2.3(i)), and ∂_0 descends to ∂ . Thus ∂ maps onto $\overline{\{0\}}$. $\square$

Lemma 3.5. *∂ is injective.*

Proof. Recall from Lemma 2.4 that, with D stable, $[C_n, SD] = \pi_1(\text{Hom}(C_n, D), 0)$ is an abelian group under concatenation. We compute in the fundamental groupoid of $\text{Hom}(C_n, D)$, writing $*$ for concatenation, $\overline{(\cdot)}$ for reversal, and const for a constant path.

Let (ζ_n) represent a class in $\ker \partial$, so that $\text{h-}\varinjlim(\underline{0}, \underline{\zeta})$ is asymptotically null-homotopic. This null-homotopy is an asymptotic morphism $C \xrightarrow{\sim} C([0, 1], D)$. By Theorem 2.3(i) it is asymptotically homotopic to $\text{h-}\varinjlim(\underline{\Phi}, \underline{H})$ for a strong homotopy morphism $(\underline{\Phi}, \underline{H}): (C_n) \rightarrow C([0, 1], D)$. By functoriality of $\text{h-}\varinjlim$ under $\text{ev}_t: C([0, 1], D) \rightarrow D$, its ends $(\text{ev}_0\Phi_n, \text{ev}_0H_n)$ and $(\text{ev}_1\Phi_n, \text{ev}_1H_n)$ are strong homotopy morphisms whose homotopy limits are asymptotically homotopic to $\text{h-}\varinjlim(\underline{0}, \underline{\zeta})$ and to 0, respectively. By Theorem 2.3(ii), these ends are then homotopic, as strong homotopy morphisms, to $(\underline{0}, \underline{\zeta})$ and to the zero morphism. Such homotopies provide, for every n ,

- a path $a_n: \text{ev}_0\Phi_n \rightarrow 0$ with $\text{ev}_0H_n * (a_{n+1} \circ \gamma_n) \simeq a_n * \zeta_n$, and
- a path $c_n: \text{ev}_1\Phi_n \rightarrow 0$ with $\text{ev}_1H_n * (c_{n+1} \circ \gamma_n) \simeq c_n$.

Indeed, each homotopy is itself a strong homotopy morphism into $C([0, 1], D)$: its maps are the paths a_n in the first case and c_n in the second, and its connecting squares are fillings whose two ways around give precisely these two relations. Writing Φ_n also for the path $s \mapsto \text{ev}_s\Phi_n$ from $\text{ev}_0\Phi_n$ to $\text{ev}_1\Phi_n$, the connecting square H_n of $(\underline{\Phi}, \underline{H})$ gives

$$(5) \quad (\text{ev}_0H_n) * (\Phi_{n+1} \circ \gamma_n) \simeq \Phi_n * (\text{ev}_1H_n).$$

Put $p_n := \overline{a_n} * \Phi_n * c_n$, a loop at 0, i.e., an element of $[C_n, SD]$. Then $p_n - \gamma_n^* p_{n+1}$ is represented by

$$\overline{a_n} * \Phi_n * c_n * \overline{(c_{n+1}\gamma_n)} * \overline{(\Phi_{n+1}\gamma_n)} * (a_{n+1}\gamma_n) \simeq \overline{a_n} * (\text{ev}_0H_n) * (a_{n+1}\gamma_n) \simeq \zeta_n.$$

The first $\simeq$ substitutes $\overline{(\Phi_{n+1}\gamma_n)} \simeq \overline{(\text{ev}_1H_n)} * \overline{\Phi_n} * (\text{ev}_0H_n)$ from Equation (5), cancels $c_n * \overline{(c_{n+1}\gamma_n)} * \overline{(\text{ev}_1H_n)} \simeq \text{const}$ (using $\text{ev}_1H_n \simeq (c_{n+1}\gamma_n) * \overline{c_n}$), and then cancels $\Phi_n * \overline{\Phi_n} \simeq \text{const}$; the second substitutes $\text{ev}_0H_n \simeq a_n * \zeta_n * (a_{n+1}\gamma_n)$. Hence $\zeta = (1 - \text{sh})(p)$ and $[(\zeta_n)] = 0$ in $\varprojlim^1[C_n, SD]$. $\square$

Proposition 3.6. Θ is surjective with $\ker \Theta = \{\overline{0}\}$, and $\partial: \varprojlim^1[C_n, SD] \rightarrow \{\overline{0}\}$ is an isomorphism whose inverse sends $[[\psi]] \in \{\overline{0}\}$ to the class of $([\Lambda_n])_n$ in $\varprojlim^1[C_n, SD]$ constructed in Lemma 3.4. Therefore, the sequence of Theorem 3.1 is exact.

Proof. Surjectivity of Θ and $\ker \Theta = \{\overline{0}\}$ follow from Proposition 2.8. By Lemma 3.3 ∂ is a well-defined homomorphism into $\{\overline{0}\}$, surjective by Lemma 3.4 and injective by Lemma 3.5. Hence it is an isomorphism onto $\{\overline{0}\}$, with inverse as stated (the class of $([\Lambda_n])_n$ is independent of all the choices made in its construction, ∂ being injective). Thus $\text{im } \partial = \{\overline{0}\} = \ker \Theta$ with ∂ injective and Θ surjective, i.e., the sequence of Theorem 3.1 is exact. $\square$

Proof of Theorems 3.1 and 1.1. Theorem 3.1 combines Lemmas 3.2–3.5 and Proposition 3.6. Theorem 1.1 is the case $C = SA \otimes \mathcal{K}$, $D = SB \otimes \mathcal{K}$ ($SD = S^2B \otimes \mathcal{K}$, $[[C, D]] = E(A, B)$), with Proposition 2.8. $\square$

4. NATURALITY AND COMPARISONS

In this final section A and B are separable C^* -algebras and (C_n, γ_n) is a shape system for $SA \otimes \mathcal{K}$. We record the naturality of the Milnor sequence, a Hausdorffness criterion and the resulting dichotomy, and the comparisons with the sequences of Willett–Yu and Schochet.

Proposition 4.1. *The outer terms of the sequence of Theorem 1.1 do not depend on the chosen shape system, and the sequence is natural in A and B .*

Proof. Theorem 1.1 identifies both outer terms with objects intrinsic to the pair (A, B) : for every shape system of $SA \otimes \mathcal{K}$, Proposition 2.8 identifies $\varprojlim[C_n, SB \otimes \mathcal{K}]$ with $EL(A, B)$, and the theorem identifies $\varprojlim^1[C_n, S^2B \otimes \mathcal{K}]$ with $\{0\}$. The sequence is natural in B by post-composition. For naturality in A , let A' be a separable C^* -algebra and (C'_n, γ'_n) a shape system for $SA' \otimes \mathcal{K}$. A $*$ -homomorphism $f: A \rightarrow A'$ induces $Sf \otimes \text{id}_{\mathcal{K}}: SA \otimes \mathcal{K} \rightarrow SA' \otimes \mathcal{K}$, represented by a strong map of systems $(C_n, \gamma_n) \rightarrow (C'_n, \gamma'_n)$, the system-to-system notion underlying the strong homotopy morphisms of Section 2 [Dad94, Definition 1.5 and Theorem 3.5]. Indeed, after reindexing, it consists of $*$ -homomorphisms $\xi_n: C_n \rightarrow C'_{m(n)}$ with homotopies filling the squares $\xi_{n+1} \circ \gamma_n \simeq \gamma'_{m(n+1), m(n)} \circ \xi_n$, compatibly with $Sf \otimes \text{id}_{\mathcal{K}}$ under the canonical maps. On homotopy classes the filled squares commute, so the maps ξ_n^* form a morphism from the reindexed towers over (C'_n) to the towers over (C_n) and induce maps on $\varprojlim$ and $\varprojlim^1$, reindexing being cofinal. The square with Θ commutes by the same compatibility, and the square with ∂ commutes because pre-composing $\text{h-}\varprojlim(\mathbb{Q}, \eta)$ with $Sf \otimes \text{id}_{\mathcal{K}}$ is again a homotopy limit, computed from the pulled-back loop data by functoriality of the homotopy limit under composition of strong maps [Dad94, Theorem 2.4]. On $\varprojlim^1$, the reindexing replaces each loop by the concatenation of the pulled-back loops over the skipped stages, which is the isomorphism induced by passage to a cofinal subsequence. $\square$

Corollary 4.2. *Let A and B be separable C^* -algebras and let (C_n, γ_n) be a shape system for $SA \otimes \mathcal{K}$, with canonical maps $\gamma_{\infty, n}$. A class $x \in E(A, B)$ lies in $\overline{\{0\}}$ if and only if $\gamma_{\infty, n}^*(x) = 0$ in $H(C_n, SB \otimes \mathcal{K})$ for every n : the phantom classes are exactly the classes that vanish on every stage of a shape system.*

Proof. By Proposition 2.8, $\ker \Theta = \overline{\{0\}}$, and the components of Θ are induced by the maps $\gamma_{\infty, n}^*$. $\square$

For the next two results, B being separable, we may assume without loss of generality that the towers at hand consist of countable groups. Indeed, by Theorem 2.2(ii) each structure map of the tower $(H(C_n, S^2B \otimes \mathcal{K}))_n$ factors through a countable set, so the groups $\Gamma_n := \text{im } \gamma_n^* \subseteq H(C_n, S^2B \otimes \mathcal{K})$ are countable. The inclusions $\Gamma_n \rightarrow H(C_n, S^2B \otimes \mathcal{K})$ and the corestrictions $H(C_{n+1}, S^2B \otimes \mathcal{K}) \rightarrow \Gamma_n$ of γ_n^* satisfy the hypotheses of Lemma 2.5 (there, take $G_n = \Gamma_n$ and $K_n = H(C_n, S^2B \otimes \mathcal{K})$, with the inclusions as the maps q_n and the corestrictions as the maps r_n), so together with Lemma 2.6 we obtain

$$(6) \quad \varprojlim^1[C_n, S^2B \otimes \mathcal{K}] \cong \varprojlim^1 \Gamma_n,$$

the $\varprojlim^1$ of a tower of countable abelian groups.

Recall that a tower (G_n, g_n) satisfies the *Mittag-Leffler condition* if for each n the descending chain of subgroups $\text{im}(G_m \rightarrow G_n) \subseteq G_n$, $m \geq n$, is eventually constant: there is $m_0 \geq n$ with $\text{im}(G_m \rightarrow G_n) = \text{im}(G_{m_0} \rightarrow G_n)$ for all $m \geq m_0$.

Corollary 4.3. *Let A and B be separable C^* -algebras and let (C_n, γ_n) be a shape system for $SA \otimes \mathcal{K}$. Then $E(A, B)$ is Hausdorff, that is $E(A, B) = EL(A, B)$, if and only if $\varprojlim^1 [C_n, S^2 B \otimes \mathcal{K}] = 0$, if and only if the tower $(\Gamma_n)_n$ of Equation (6) is Mittag-Leffler. In particular $E(A, B)$ is Hausdorff whenever the tower $([C_n, S^2 B \otimes \mathcal{K}])_n$ is Mittag-Leffler.*

Proof. The first equivalence is Theorem 1.1, $E(A, B)$ being Hausdorff exactly when $\overline{\{0\}} = 0$. A Mittag-Leffler tower of abelian groups has vanishing $\varprojlim^1$, and for towers of countable abelian groups the converse holds as well [Sch03, Theorem 3.11]. By Equation (6) this gives the second equivalence. For the final statement: if the tower $([C_n, S^2 B \otimes \mathcal{K}])_n$ is Mittag-Leffler, then its $\varprojlim^1$ vanishes, and by Theorem 1.1 so does $\overline{\{0\}}$. $\square$

Remark 4.4. The $\varprojlim^1$ of a tower of countable abelian groups is either trivial or uncountable, by an argument of Gray [Gra66, p. 242] (see also [Sch03, Proposition 3.5]). By Equation (6), $\overline{\{0\}}$ is therefore either trivial or uncountable: if $E(A, B)$ is not Hausdorff, it contains uncountably many phantom classes. This is the E -theoretic counterpart of [WY26, Corollary 7.9], where the observation is credited to Schochet.

Proposition 4.5 (Comparison with the Willett–Yu sequence). *Let A and B be separable C^* -algebras with A nuclear, and let (C_n, γ_n) be a shape system for $SA \otimes \mathcal{K}$. The isomorphism $E(A, B) \cong KK(A, B)$, an isomorphism of topological groups for nuclear A , carries $\overline{\{0\}}$ onto $\overline{\{0\}}_{KK(A, B)}$ and induces $EL(A, B) \cong KL(A, B)$, so Theorem 1.1 becomes the sequence*

$$(7) \quad 0 \longrightarrow \overline{\{0\}}_{KK(A, B)} \longrightarrow KK(A, B) \longrightarrow KL(A, B) \longrightarrow 0.$$

The Willett–Yu controlled- KK Milnor sequence is isomorphic to it as an extension of $KL(A, B)$ by $\overline{\{0\}}_{KK(A, B)}$; in particular

$$\varprojlim^1 [C_n, S^2 B \otimes \mathcal{K}] \cong \overline{\{0\}}_{KK(A, B)} \cong \varprojlim^1 KK_{\varepsilon_n}(X_n, SB),$$

both being the kernel of $KK(A, B) \rightarrow KL(A, B)$. Here (X_n, ε_n) is a cofinal sequence of finite subsets $X_n \subseteq A$ and tolerances ε_n for the controlled KK -groups of [WY26, Sections 6 and 7].

Proof. For A nuclear the canonical natural transformation $KK(A, B) \rightarrow E(A, B)$ is an isomorphism [CH90, Corollary 9]. It carries $\overline{\{0\}}_{KK(A, B)}$ onto $\overline{\{0\}}$ and induces $KL(A, B) \cong EL(A, B)$ [CS24, Section 4.2]. Under these identifications the sequence of Theorem 1.1 becomes the extension (7), with $\varprojlim^1 [C_n, S^2 B \otimes \mathcal{K}] \cong \overline{\{0\}}_{KK(A, B)}$, $\varprojlim [C_n, SB \otimes \mathcal{K}] \cong KL(A, B)$, and Θ the canonical surjection $KK(A, B) \rightarrow KL(A, B)$. Willett–Yu prove the same two identifications for their controlled tower $(\varprojlim KK_{\varepsilon_n}(X_n, B) \cong KL(A, B)$ and $\varprojlim^1 KK_{\varepsilon_n}(X_n, SB) \cong \overline{\{0\}}_{KK(A, B)})$, compatibly with the same surjection [WY26, Theorems 6.14 and 7.8]. Then the two Milnor sequences are isomorphic as extensions and their $\varprojlim^1$ terms agree. $\square$

The Willett–Yu sequence, like ours, needs no nuclearity: their identifications hold for arbitrary separable A and B [WY26, Theorems 6.14 and 7.8]. The hypothesis

in Proposition 4.5 serves only to identify $E(A, B)$ with $KK(A, B)$ (and for this much less is needed, see e.g. [Bla98, Theorem 25.6.3]). In general, the two sequences describe the closures of zero of potentially different groups.

Specialized to E -theory, the sequence of Bunke and Dünzinger presents the same canonical extension $0 \rightarrow \overline{\{0\}} \rightarrow E(A, B) \rightarrow EL(A, B) \rightarrow 0$ as Theorem 1.1, with the outer terms computed from the E -theory groups of the stages of a shape system of $S^2A \otimes \mathcal{K}$ rather than from homotopy classes of $*$ -homomorphisms [BD24, Section 4]. The identification of its $\varprojlim^1$ term with $\overline{\{0\}}$ there rests on the ∞ -categorical framework of that paper, which is not used here.

Proposition 4.6 (The UCT case). *Let A and B be separable C^* -algebras, suppose A satisfies the UCT, and let (C_n, γ_n) be a shape system for $SA \otimes \mathcal{K}$. Then*

$$\varprojlim^1 [C_n, S^2B \otimes \mathcal{K}] \cong \text{Pext}_{\mathbb{Z}}^1(K_{*+1}(A), K_*(B))$$

and $\varprojlim [C_n, SB \otimes \mathcal{K}] \cong \text{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B))$.

Proof. A separable C^* -algebra satisfies the UCT for all B if and only if it is KK -equivalent to a commutative C^* -algebra, by the argument of [RS87, Corollary 7.5] (cf. the discussion following Remark 7.6 there). The canonical natural transformation $KK \rightarrow E$ (see [CH90, Corollary 9] or [Bla98, Corollary 25.5.8]) is compatible with the composition products, so it carries KK -equivalences to E -equivalences. Hence A is E -equivalent to a commutative C^* -algebra. For such A , it is shown in [CS24, Section 4.2 and Theorem 4.8] that $\overline{\{0\}}$ is the image of the subgroup of pure extensions under the injective map $\text{Ext}_{\mathbb{Z}}^1(K_{*+1}(A), K_*(B)) \rightarrow E(A, B)$ of the E -theoretic UCT, and that $EL(A, B) \cong \text{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B))$. The statement now follows from Theorem 1.1 and Proposition 2.8. $\square$

Besides the bootstrap class of [RS87], the hypothesis of Proposition 4.6 covers non-nuclear algebras. A free group $\mathbb{F}_n$ is a-T-menable, so its reduced group C^* -algebra $C_r^*(\mathbb{F}_n)$ satisfies the UCT [Tu99, Theorem 9.3 and Proposition 10.7]. For $n \geq 2$ this algebra is not nuclear.

Acknowledgments. The author is grateful to Christopher Schafhauser for many discussions about the topology on E -theory, and to Rufus Willett for enlightening conversations about controlled KK -theory and its Milnor exact sequence.

REFERENCES

- [BD24] Ulrich Bunke and Benjamin Dünzinger, *E-theory is compactly assembled*, arXiv:2402.18228, 2024.
- [BK72] Aldridge K. Bousfield and Daniel M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, vol. 304, Springer-Verlag, Berlin-New York, 1972. MR 365573
- [Bla85] Bruce Blackadar, *Shape theory for C^* -algebras*, Math. Scand. **56** (1985), no. 2, 249–275. MR 813640
- [Bla98] ———, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998. MR 1656031
- [Bro75] Lawrence G. Brown, *Operator algebras and algebraic K-theory*, Bull. Amer. Math. Soc. **81** (1975), no. 6, 1119–1121. MR 0383090
- [Bro77] ———, *Characterizing $\text{Ext}(X)$, K-theory and operator algebras* (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), Springer, Berlin, 1977, pp. 10–18. Lecture Notes in Math., Vol. 575. MR 0474275

- [Bro84] ———, *The universal coefficient theorem for Ext and quasidiagonality*, Operator algebras and group representations, Vol. I (Neptun, 1980), Monogr. Stud. Math., vol. 17, Pitman, Boston, MA, 1984, pp. 60–64. MR 731763
- [CH90] Alain Connes and Nigel Higson, *Déformations, morphismes asymptotiques et K -théorie bivariante*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 2, 101–106. MR 1065438
- [CS24] José R. Carrión and Christopher Schafhauser, *A topology on E -theory*, J. Lond. Math. Soc. (2) **109** (2024), no. 6, Paper No. e12917, 32. MR 4751861
- [Dad94] Marius Dadarlat, *Shape theory and asymptotic morphisms for C^* -algebras*, Duke Math. J. **73** (1994), no. 3, 687–711. MR 1262931
- [Dad05] ———, *On the topology of the Kasparov groups and its applications*, J. Funct. Anal. **228** (2005), no. 2, 394–418. MR 2175412
- [GHT00] Erik Guentner, Nigel Higson, and Jody Trout, *Equivariant E -theory for C^* -algebras*, Mem. Amer. Math. Soc. **148** (2000), no. 703, viii+86. MR 1711324
- [Gra66] Brayton I. Gray, *Spaces of the same n -type, for all n* , Topology **5** (1966), 241–243. MR 196743
- [Rø95] Mikael Rørdam, *Classification of certain infinite simple C^* -algebras*, J. Funct. Anal. **131** (1995), no. 2, 415–458. MR 1345038
- [RS87] Jonathan Rosenberg and Claude L. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov’s generalized K -functor*, Duke Math. J. **55** (1987), no. 2, 431–474. MR 894590
- [Sch96] Claude L. Schochet, *The UCT, the Milnor sequence, and a canonical decomposition of the Kasparov groups*, K -Theory **10** (1996), no. 1, 49–72. MR 1373818
- [Sch98] ———, *Correction to: “The UCT, the Milnor sequence, and a canonical decomposition of the Kasparov groups” [K -Theory **10** (1996), no. 1, 49–72; MR1373818 (97d:46088)]*, K -Theory **14** (1998), no. 2, 197–199. MR 1628275
- [Sch01] ———, *The fine structure of the Kasparov groups. I. Continuity of the KK -pairing*, J. Funct. Anal. **186** (2001), no. 1, 25–61. MR 1863291
- [Sch02] ———, *The fine structure of the Kasparov groups. II. Topologizing the UCT*, J. Funct. Anal. **194** (2002), no. 2, 263–287. MR 1934604
- [Sch03] ———, *A Pext primer: pure extensions and $\lim^1$ for infinite abelian groups*, New York Journal of Mathematics. NYJM Monographs, vol. 1, State University of New York, University at Albany, Albany, NY, 2003, Available electronically at <http://nyjm.albany.edu/m/2003/1v.pdf>. MR 1993677
- [Tu99] Jean-Louis Tu, *La conjecture de Baum-Connes pour les feuilletages moyennables*, K -Theory **17** (1999), no. 3, 215–264. MR 1703305
- [WY24] Rufus Willett and Guoliang Yu, *The universal coefficient theorem for C^* -algebras with finite complexity*, Mem. Eur. Math. Soc., vol. 8, EMS Press, Berlin, 2024. MR 4711665
- [WY26] ———, *Controlled KK -theory and a Milnor exact sequence*, Doc. Math. **31** (2026), no. 3, 503–581. MR 5061886

DEPARTMENT OF MATHEMATICS, TEXAS CHRISTIAN UNIVERSITY, FORT WORTH, TX 76129, USA
 Email address: j.carrión@tcu.edu