NSF/CBMS REGIONAL CONFERENCE IN THE MATHEMATICAL SCIENCES: CLASSIFYING AMENABLE OPERATOR ALGEBRAS JUNE 2–6, 2025

DESCRIPTION OF LECTURES

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Lecture 1: The classification theorem

The lecture will be devoted to a discussion of the following theorem:

Theorem A (The classification theorem: the unital case)**.** *Let A and B be simple, separable, nuclear, regular C^*-algebras satisfying the UCT. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$. Moreover any isomorphism between $\mathrm{Ell}(A)$ and $\mathrm{Ell}(B)$ is induced by an isomorphism between A and B.*

The theorem is the culmination of several decades of work spread over thousands of pages in the literature with at least a few dozen authors directly involved in the proof over this time. The theorem will be compared to the fundamental result of Connes and Haagerup that simple, separably acting, injective von Neumann algebras are classified by their type and flow of weights. In the von Neumann algebraic setting, the regularity condition is automatic. There is also no clear analogy of the UCT condition—it appears in the C^*-algebraic setting due to the necessity of K-theoretic invariants. Some applications will also be discussed: for example, as the range of the Elliott invariant is well understood, the classification theorem shows all such C^*-algebras are (twisted) groupoid C^*-algebras and admit Cartan subalgebras.

The participants will be familiar with the first three hypotheses, but many will be less familiar with the latter two. A detailed discussion of regularity and the UCT will be deferred to Lectures 5 and 3, respectively. For this first lecture, the hypotheses will be explained through examples, focusing on the class of crossed product C^*-algebras $C(X) \rtimes G$ for a compact metrizable space X, a discrete group G, and an action $G \curvearrowright X$ by homeomorphisms. I will discuss sufficient (and in some cases, necessary) conditions on the action which guarantee the crossed product C^*-algebras is classifiable: briefly, this holds if the action is free and minimal, X is finite dimensional, and G is elementary amenable. I will also discuss how the invariant relates to the dynamics, restricting to the case of the irrational rotation algebras $C(\mathbb{T}) \rtimes \mathbb{Z}$, given by an aperiodic rotation of the circle.

Lecture 2: Intertwining Arguments

The most basic tools underlying classification results in C^*-algebras are intertwining arguments. For unital, separable C^*-algebras A and B and unital morphisms $\phi, \psi\colon A \to B$, we say ϕ and ψ are *approximately unitarily equivalent* if there is a sequences of unitaries in B which in the limit, pointwise in norm, conjugate ϕ to ψ. A fundamental observation of Elliott is that if there are morphisms $A \to B$ and $B \to A$ which are inverses up to approximate unitary equivalence, then $A \cong B$. This has become known as *Elliott's two-sided intertwining argument*. Although the result is not too difficult to prove, it leads to a template for classifying C^*-algebras up to isomorphism: one first classifies morphisms up to approximate unitary equivalence. This is the strategy taken in the classification theorem.

I will lead into this result with a discussion and sketch of proof of Elliott's classification of AF-algebras: those C^*-algebras which are the closure of a union of an increasing sequence of finite dimensional C^*-subalgebras. The basic idea is that morphisms between finite dimensional C^*-algebras are completely understood by the Wedderburn theorems, and hence the entire content of the classification is the intertwining arguments. AF-algebras and their classification will be familiar to a large portion of the audience, but the presentation will be somewhat different than the standard textbook presentations in that it will isolate and highlight the intertwining arguments and will focus on a one-sided version classifying morphisms from AF-algebras to very general codomains.

This also leads into another fundamental intertwining argument. In the case of AF-algebras, one classifies morphisms out of finite dimensional C^*-algebras and uses a construction known as "Elliott's one-sided intertwining argument" to extend the classification result to the inductive limit. Without an inductive limit decomposition of the domain (say in the setting of Theorem A), one replaces morphisms out of subalgebras with *approximate morphisms*: sequences of linear maps $A \to B$ which preserve multiplication in the limit. An abstract version of the one-sided intertwining argument reduces classifying morphisms to classifying approximate morphisms. Or equivalently, the classification reduces to classifying morphisms from a C^*-algebra A to the *sequence algebra* $B_\infty = \ell^\infty(B)/c_0(B)$ of a C^*-algebra B. This will reappear in Lecture 6.

Lecture 3: KK-Theory and the UCT

Kasparov's KK-theory is a formidable tool in C^*-algebra theory. It provides a link between (and a simultaneous generalization of) operator K-theory and the K-homology/extension theory of Atiyah and Brown–Douglas–Fillmore. Very loosely speaking, KK-theory can be viewed as an abelianization of the category of C^*-algebras with homotopy classes of *-homomorphisms. It serves as a bridge between C^*-algebras and their K-theory groups. For instance, every morphisms of C^*-algebras $A \to B$ induces an element of $KK(A, B)$, and every element of $KK(A, B)$ induces a morphism $K_*(A) \to K_*(B)$. The computational properties of KK-theory allow for manipulations of elements of $KK(A, B)$ which are not possible at the level of *-homomorphisms. It is impossible to give a full description of KK-theory in a 50 minute lecture, but I will give an intuitive description, highlighting the definition in terms of Cuntz pairs, which is most relevant ot the the classification theory and, in particular, to the next several lectures.

This lecture will also discuss the *Universal Coefficient Theorem (UCT)* which essentially states that KK-theory is completely determined by its action on the operator K-theory functors K_*. In the context of Theorem A, if two such C^*-algebras A and B have isomorphic Elliott invariants, then, in particular, they have isomorphic K-theory groups. The UCT allows one to conclude that A and B are *KK-equivalent* in the sense that there are elements in $KK(A, B)$ and $KK(B, A)$ which are inverses under Kasparov product (the natural notion of composition in KK-theory). This is in some sense the starting point for obtaining an isomorphism $A \to B$.

Lecture 4: Non-stable extension theory

The main drawback of KK-theory and extension theory is that one loses a lot of information in the stabilizations and homotopy equivalences needed to obtain computational tools. Non-stable extension theory focuses on removing these stabilizations and replacing the homotopies with more rigid equivalence relations (e.g. approximate unitary equivalence). A landmark result in this direction is the Kirchberg–Phillips theorem which, in one form, states that for a separable nuclear C^*-algebra A and simple, non-unital C^*-algebra B, the group

$KK(A, B)$ is naturally in bijection with asymptotic unitary equivalence classes of *-homomorphisms $A \to B \otimes \mathcal{O}_\infty$. For the stably finite classification, the main underlying tool from non-stable KK-theory is the Dadarlat–Eilers–Lin *stable uniqueness theorem*.

Roughly, suppose A and B are separable C^*-algebras. Let $\mathcal{K}$ denote the compact operators on a separable, infinite dimensional Hilbert space and let $M(B \otimes \mathcal{K})$ be the stable multiplier algebra of B. Two *-homomorphisms $\phi, \psi \colon A \to M(B \otimes \mathcal{K})$ with $\phi(a) - \psi(a) \in B \otimes \mathcal{K}$ induce an element $[\phi, \psi] \in KK(A, B)$—this will be the picture of $KK(A, B)$ taken as the definition in Lecture 3. The stable uniqueness theorem states that if $[\phi, \psi] = 0$, there is a *-homomorphism $\theta \colon A \to M(B \otimes \mathcal{K})$ and a one-parameter family of unitaries (u_t) in the minimal unitization of $M_2(B \otimes \mathcal{K})$, which in the limit conjugates $\phi \oplus \theta$ to $\psi \oplus \theta$ pointwise in norm. The point of the theorem is that the stabilizations in KK-theory can be collected in a single map θ (and there is some freedom in choosing θ), and the homotopies can be implemented by unitaries.

The lecture will focus around the notion of *absorption*, which is an abstracted version of the conclusion of Voiculescu's theorem. When ϕ and ψ are absorbing, one can take the summand θ to be either ϕ or ψ, and by doing both of these, one can conclude that $\phi \oplus \phi$ and $\psi \oplus \psi$ are asymptotically unitarily equivalent via unitaries in the minimal unitization of $M_2(B \otimes \mathcal{K})$. A slightly more refined version allows one to replace the 2×2 matrix amplification with a tensor factor of $\mathcal{Z}$ to get a similar equivalence between $\phi \otimes 1_{\mathcal{Z}}$ and $\psi \otimes 1_{\mathcal{Z}}$.

The proofs are somewhat involved. Instead of going into the details of the result (or even a precise statement), the goal of the lecture will be to introduce the notion of absorption, drawing intuition from Voiculescu's theorem, and to highlight the fundamental role in plays in passing from KK-theoretic computations back to C^*-algebraic results. I will end with a statement of the Elliott–Kucerovsky theorem, which gives a very concrete method for verifying absorption under nuclearity hypotheses. Using this theorem, the absorption will be accessed in Lecture 6 using the regularity theory laid out in Lecture 5.

Lecture 5: $\mathcal{Z}$-stability and regularity

This lecture will be devoted to the remaining hypothesis in Theorem A: regularity. The three main flavors of regularity are

(i) *finite nuclear dimension*, a non-commutative analogue of a topological space being finite dimensional,

(ii) *$\mathcal{Z}$-stability*, an analytic condition characterized by a rich supply of approximately central sequences and analogous to McDuff's property in von Neumann algebras, and
(iii) *strict comparison*, asking that the comparison theory in the C^*-algebra is determined by scalar-valued rank functions, analogous to Murray and von Neumann's classification of projections in a finite factor by their trace.

The *Toms–Winter Conjecture* (circa 2005) states that the three conditions are equivalent for unital, separable, simple, nuclear, non-elementary C^*-algebras. The implications (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii) are known and the remaining implication (iii)$\Rightarrow$(ii) is known in several cases of interest (e.g. for C^*-algebras with unique trace).

The regularity conditions (ii) and (iii) will be most relevant to the later lectures, and this is where the focus will be (although (i) is often how regularity is verified in examples). I will define these conditions, and in particular, sketch a construction of the Jiang–Su algebra $\mathcal{Z}$. I will also discuss the connection between tensorial absorption of $\mathcal{Z}$ and the central sequence algebras, drawing analogies with McDuff's analogous results from von Neumann algebra theory.

Lecture 6: The trace-kernel extension

The trace-kernel extension lies at the heart of the new approach to the classification theorem. Let A and B be as in Theorem A and assume they have at least one trace (the traceless case of Theorem A follows from the Kirchberg–Phillips theorem mentioned in Lecture 4). The non-stable extension theory cannot apply to such C^*-algebras directly as simple C^*-algebras, by definition, have no ideals. However, the sequence algebra $B_\infty = \ell^\infty(B)/c_0(B)$ discussed in Lecture 2 has a very natural ideal $J_B \trianglelefteq B_\infty$ consisting the "tracially null sequences" in B. This is the *trace-kernel ideal* of B. The quotient $B^\infty = B_\infty/J_B$ carries natural von Neumann algebraic structure—loosely speaking, it is a bundle of II_1 factors. Making use of the central sequences in B^∞ arising from $\mathcal{Z}$-stability, Connes's theorem can be combined with a partition of unity argument to classify morphisms $A \to B^\infty$ up to unitary equivalence by tracial data. This reduces the classification of maps $A \to B_\infty$ to classifying lifts along the trace-kernel extension as pictured below:

$$\begin{array}{ccccccccc} & & & & & & A & & \\ & & & & & {\overset{?}{\swarrow}} & \big\downarrow{\scriptstyle\theta} & & \\ 0 & \longrightarrow & J_B & \xrightarrow{j_B} & B_\infty & \xrightarrow{q_B} & B^\infty & \longrightarrow & 0. \end{array}$$

Extension theoretic regularity coming from strict comparison of B allows one to reduce to classify such lifts.

After recalling the definition of the sequence algebra of B_∞ and defining the trace-kernel extension, this lecture will focus on the case when B has unique trace. In this case, the partition of unity argument using $\mathcal{Z}$-stability can be avoided entirely. I will also only focus on the question of when there is a lift $A \to B_\infty$ of a given map θ and defer discussion the uniqueness problem for such lifts to Lecture 7. This special case is enough to prove the celebrated *quasidiagonality theorem* of Tikuisis, White, and Winter, and the lecture will be focused around this application. In the larger picture, this provides a method for constructing an approximate morphism $A \to B$, which is the first step towards producing an isomorphism, as discussed in Lecture 2 on intertwining arguments. Lectures 7 and 8 will refine this idea, leading to the full classification theorem.

Lecture 7: Classification of lifts

This will be a continuation of Lecture 6 describing all lifts of a given map $\theta\colon A \to B^\infty$ up to unitary equivalence. Strict comparison of B is used to obtain a refined non-stable extension theory through the Elliott–Kucerovsky theorem. The non-stable extension theory from lecture 4 and KK-theory from lecture 3 will be recalled and refined to discuss how to classify morphisms $A \to B_\infty$ in terms of their behavior on traces and a secondary invariant in the group $KK(A, J_B)$.

Lecture 8: Computing the invariant

The intertwining arguments of Lecture 2 reduce the Theorem A to the classification of morphisms $A \to B_\infty$. Then Lecture 7 obtains the classification of such morphisms in terms of traces and $KK(A, J_B)$. The goal of this final lecture on Theorem A will be to relate the group $KK(A, J_B)$ to K-theoretic data of B_∞. Roughly, $KK(A, J_B)$ can be computed in terms of the K-theory groups $K_*(B_\infty; \mathbb{Z}/n)$ and (a variation of) the algebraic K_1-group $\bar{K}_1^{\mathrm{alg}}(B_\infty)$. This computation then readily implies Theorem A from Lecture 1 via the intertwining arguments from Lecture 2. In fact, one obtains a stronger version of Theorem A which classifies embeddings $A \to B$ under the same hypotheses (and even somewhat more generally).

I will state the result of the computation in general but will restrict all further discussion to the case B has unique trace with divisible K-theory groups, which drastically simplifies the computation of $KK(A, J_B)$ and only involves working with the more familiar K-theory groups K_0 and K_1. This special case is also enough to prove separable, nuclear, monotracial C^*-algebras with a unique trace can be embedded into a simple AF-algebra.

Lecture 9: Equivariant classification I

The final two lectures are devoted to the next steps in the classification theory. Following Connes's fundamental work on injective factors, there was a large body of work of Connes, Jones, and Ocneanu in the subsequent decade on the structure of symmetries of such algebras. A high point of this investigation was Ocneanu's theorem that every countable, discrete, amenable group admits an outer action on the separably acting, hyperfinite II_1 factor and this action is unique up to cocycle conjugacy. The classification and structure of group actions on simple, nuclear C^*-algebras is still in early stages but has been a rapidly growing subject.

This first lecture on equivariant classification will put the problem in context and give basic definitions, such as the definition of cocycle conjugacy and the relevance of this equivalence relation. I will also discuss the notion of cocycle morphisms between group actions and establish basic properties of these morphism. The naive notion of morphism between group actions is a morphism of the underlying algebras which intertwines the action. For technical reasons, this notion of morphism is too restrictive. Essentially, the intertwining arguments of Lecture 2 suggests one should study morphisms up to (approximately) inner automorphisms of the codomain, and there are very few inner automorphisms in the category of equivariant morphisms. The notion of a cocycle morphism relaxes the equivariance condition requiring that a morphism preserves the group action only up to a 1-cocycle, and this weaker notion of morphisms leads to an abundance of inner automorphisms. Further, cocycle conjugacy is exactly isomorphism in the category of group actions and cocycle morphisms.

The definitions of cocycle morphisms and related concepts look a bit mysterious when they are seen for the first time. The goal of the lecture will be to present this material at a leisurely pace and show how the notion arises naturally, and to explain the basic operational constructions such as composition and conjugacy. The lecture will conclude with equivariant versions of the intertwining arguments from Lecture 2 and a brief discussion of a strategy for obtaining general equivariant classification results by incorporating group actions in the material from the first 8 lectures. Only a few of these modifications have been done, such as the intertwining arguments, building an equivariant extension theory and KK-theory, and very recently, some preliminary results on a non-stable KK-theory for group actions. Fleshing out equivariant versions of the various components of the proof of the classification theorem is a largely unexplored topic which will be important as the equivariant theory further develops.

Lecture 10: Equivariant Classification II

This final lecture will be devoted to a consequence of Theorem A which classifies actions of finite groups on C^*-algebras in Theorem A under the assumption that the group actions satisfy the Rokhlin property. The proof is essentially due to Izumi (circa 2004), where the result was proved in the purely infinite setting. The only modification to Izumi's proof is quoting the classification of automorphisms of the algebras in Theorem A (discussed in Lecture 8) in place of the Kirchberg–Phillips theorem. While the proof is now classical, it is not very well known. The Rokhlin condition on the actions is fairly restrictive, but the proof under this strong condition is highly digestible and it highlights some of the main techniques in the equivariant classification and structure theory: the intertwining arguments and a method for averaging non-equivariant classification over Følner sets in the group (or the whole group in the case here of a finite group). The latter technique is deeply hidden in most equivariant classification results, but under the extra orthogonality provided by the Rokhlin property, this step of the proof can be made very transparent. This also leads to a very definitive equivariant classification results to end the lecture series.